

# WHITTAKER PATTERNS IN THE GEOMETRY OF MODULI SPACES OF BUNDLES ON CURVES

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## INTRODUCTION

Whittaker functions are ubiquitous in the modern theory of automorphic functions. In this paper we develop a geometric version of the local Whittaker theory in the non-archimedean case. By this we mean finding appropriate Whittaker sheaves, such that the functions obtained by taking the traces of the Frobenius on the stalks are the local Whittaker functions. The desire to construct such sheaves is motivated by the geometric Langlands correspondence (see [6, 14, 15, 9]). Since Whittaker functions never have compact support, we cannot obtain them from sheaves on finite-dimensional varieties in an obvious way, and this makes the problem of constructing Whittaker sheaves non-trivial. Our approach is to use global methods, namely, the Drinfeld compactification of the moduli stack of  $N$ -bundles.

More concretely, let  $G$  be a (split connected) reductive group over  $\mathbb{F}_q$ , and  $N$  be its maximal unipotent subgroup. Consider the group  $G(\hat{\mathcal{K}})$  over the local field  $\hat{\mathcal{K}} = \mathbb{F}_q((t))$ , and its maximal compact subgroup  $G(\hat{\mathcal{O}})$ , where  $\hat{\mathcal{O}} = \mathbb{F}_q[[t]]$ . The quotient  $\text{Gr} = G(\hat{\mathcal{K}})/G(\hat{\mathcal{O}})$  can be given a structure of an ind-scheme over  $\mathbb{F}_q$ , which is called the affine Grassmannian of  $G$ . Unramified Whittaker functions give rise to functions on  $\text{Gr}$ , which are  $N(\hat{\mathcal{K}})$ -equivariant against a non-degenerate character  $\chi$ . According to the general philosophy of “faisceaux–fonctions” correspondence, these functions should have geometric counterparts as perverse sheaves on  $\text{Gr}$ . The geometric counterparts of the Whittaker functions should be sheaves on  $\text{Gr}$ , which are  $N(\hat{\mathcal{K}})$ -equivariant against  $\chi$ . It is natural to expect that the appropriately defined category of such sheaves should reflect the properties of the Whittaker functions. Unfortunately, the  $N(\hat{\mathcal{K}})$ -orbits in  $\text{Gr}$  are infinite-dimensional and so there is no obvious way to define this category on  $\text{Gr}$ .

On the other hand, let  $\text{Bun}_N$  be the algebraic stack classifying  $N$ -bundles on a smooth projective curve  $X$  over  $\mathbb{F}_q$ . V. Drinfeld has introduced a remarkable partial compactification  $\overline{\text{Bun}}_N$  of  $\text{Bun}_N$ . In this paper we argue that an appropriate “Whittaker category” can (and perhaps should) be defined on  $\overline{\text{Bun}}_N$  instead of  $\text{Gr}$ . In fact,  $\overline{\text{Bun}}_N$  can be viewed as a suitable “globalization” of the closure of a single  $N(\hat{\mathcal{K}})$ -orbit in  $\text{Gr}$ . Moreover,  $\overline{\text{Bun}}_N$  has many advantages over  $\text{Gr}$  (see Sect. 1.2.3).

In this paper we define the appropriate “Whittaker category” of perverse sheaves on  $\overline{\text{Bun}}_N$  (more precisely, on its generalization  ${}_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$ ) and prove that it does indeed possess all the properties that one would expect from it by analogy with the Whittaker

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functions. Namely, this category is semi-simple, and each irreducible object of this category is a local system on a single stratum extended by zero (these strata are the analogues of the  $N(\hat{\mathcal{K}})$ -orbits in  $\text{Gr}$ ).

As an application of the main theorems of this paper, we compute the cohomology of certain sheaves on the intersections of  $G(\hat{\mathcal{O}})$ - and  $N(\hat{\mathcal{K}})$ -orbits in  $\text{Gr}$ . Using this result we prove our conjecture from [9] in the case of a general reductive group  $G$ . In the case of  $G = GL_n$ , B.C. Ngo [19] has earlier given an elegant proof of this conjecture using a different method.<sup>1</sup> As explained in [9], the proof of this conjecture gives us a purely geometric proof of the Casselman-Shalika formula [5, 22] for the Whittaker function.

More details on the background and motivations for the present work, as well as the description of the structure of this paper, can be found in Sect. 1.

We note that the results of this paper are valid in the characteristic 0 case, when one works in the context of  $\mathcal{D}$ -modules.

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## 1. BACKGROUND AND OVERVIEW

### 1.1. Hecke algebra and Whittaker functions.

1.1.1. Let  $G$  be a split connected reductive group over  $\mathbb{F}_q$ ,  $G(\hat{\mathcal{K}})$  the corresponding group over the local field  $\hat{\mathcal{K}} = \mathbb{F}_q((t))$ , and  $G(\hat{\mathcal{O}})$ , its maximal compact group, where  $\hat{\mathcal{O}} = \mathbb{F}_q[[t]]$ . We fix a Haar measure on  $G(\hat{\mathcal{K}})$  so that  $G(\hat{\mathcal{O}})$  has measure 1.

Consider the *Hecke algebra*  $\mathbf{H}$  of  $G(\hat{\mathcal{K}})$  with respect to  $G(\hat{\mathcal{O}})$ , i.e., the algebra of  $\overline{\mathbb{Q}}_\ell$ -valued compactly supported  $G(\hat{\mathcal{O}})$ -bi-invariant functions on  $G(\hat{\mathcal{K}})$  with the convolution product:

$$(1.1) \quad (h_1 \star h_2)(g) = \int_{G(\hat{\mathcal{K}})} h_1(x) h_2(g \cdot x^{-1}) dx.$$

It is well-known that there is a bijection between the double quotient of  $G(\hat{\mathcal{K}})$  with respect to  $G(\hat{\mathcal{O}})$  and the set  $\Lambda^{++}$  of dominant coweights of  $G$ : to  $\lambda \in \Lambda^{++}$  we attach the coset of  $\lambda(t) \in T(\hat{\mathcal{K}}) \subset G(\hat{\mathcal{K}})$ , where  $T$  is the Cartan subgroup of  $G$ . Hence we obtain a basis  $\{c_\lambda\}_{\lambda \in \Lambda^{++}}$  of  $\mathbf{H}$  consisting of the corresponding characteristic functions.

It turns out that  $\mathbf{H}$  has a “better” basis. Let  ${}^L G$  be the Langlands dual group of  $G$  and let  $\text{Rep}({}^L G)$  denote the Grothendieck ring (over  $\overline{\mathbb{Q}}_\ell$ ) of its finite-dimensional representations.

The following statement is often referred to as the Satake isomorphism:

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<sup>1</sup>In the course of writing this paper we learned from B.C. Ngo that he and P. Polo obtained an independent proof of the conjecture for a general reductive group  $G$ .

*Note added in Nov 2000:* the paper by B.C. Ngo and P. Polo, *Résolutions de Demazure affines et formule de Casselman-Shalika géométriques*, has appeared as math.AG/0005022.

1.1.2. **Theorem.** ([21, 12, 11]) *There exists a canonical isomorphism of algebras  $\mathbf{H} \simeq \text{Rep}({}^L G)$ .*

1.1.3. To an element  $\lambda \in \Lambda^{++}$  we can attach the class of the corresponding irreducible  ${}^L G$ -module  $V^\lambda$  in  $\text{Rep}({}^L G)$ . Their images under the Satake isomorphism, denoted by  $A_\lambda$ , yield the “better” basis for  $\mathbf{H}$ .

Moreover, the  $A_\lambda$  have the following form:

$$(1.2) \quad A_\lambda = q^{-\langle \lambda, \check{\rho} \rangle} \left( c_\lambda + \sum_{\mu \in \Lambda^{++}; \mu < \lambda} p_{\lambda\mu} \cdot c_\mu \right), \quad p_{\lambda\mu} \in \mathbb{Z}[q].$$

1.1.4. Let  $N \subset G$  be a maximal unipotent subgroup. We define a character  $\chi : N(\hat{\mathcal{K}}) \rightarrow \mathbb{F}_q$  as follows:

$$\chi(u) = \sum_{i=1}^{\ell} \psi(\text{Res}(u_i)),$$

where  $u_i, i = 1 \dots, \ell = \dim N/[N, N]$  are natural coordinates on  $N/[N, N]$  corresponding to the simple roots,  $\text{Res} : \hat{\mathcal{K}} \rightarrow \mathbb{F}_q$  is the map  $\text{Res}(\sum_{n \in \mathbb{Z}} f_i t^i) = f_{-1}$  and  $\psi : \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_\ell^\times$  is a fixed non-trivial character.

We define the space  $\mathbf{W}$  consisting of functions  $f : G(\hat{\mathcal{K}}) \rightarrow \overline{\mathbb{Q}}_\ell$ , such that

- $f(g \cdot x) = f(g)$ , if  $x \in G(\hat{\mathcal{O}})$ ,
- $f(n \cdot g) = \chi(n) \cdot f(g)$ , if  $n \in N(\hat{\mathcal{K}})$ ,
- $f$  is compactly supported modulo  $N(\hat{\mathcal{K}})$ .

Clearly, the space  $\mathbf{W}$  is an  $\mathbf{H}$ -module with respect to the following action:

$$(1.3) \quad h \in \mathbf{H}, f \in \mathbf{W} \longrightarrow (f \star h)(g) = \int_{G(\hat{\mathcal{K}})} f(g \cdot x^{-1}) h(x) dx.$$

We call  $\mathbf{W}$  the *Whittaker module*.

There is a bijection between the double cosets  $N(\hat{\mathcal{K}}) \backslash G(\hat{\mathcal{K}}) / G(\hat{\mathcal{O}})$  and the coweight lattice  $\Lambda : \lambda \rightarrow N(\hat{\mathcal{K}}) \cdot \lambda(t) \cdot G(\hat{\mathcal{O}})$ . However, it is easy to see that if  $\lambda \in \Lambda$  is non-dominant, then  $f(\lambda(t)) = 0$  for any  $f \in \mathbf{W}$ . Therefore  $\mathbf{W}$  has a basis  $\{\phi_\lambda\}_{\lambda \in \Lambda^{++}}$ , where  $\phi_\lambda$  is the unique function in  $\mathbf{W}$  which is non-zero only on the double coset  $N(\hat{\mathcal{K}}) \cdot \lambda(t) \cdot G(\hat{\mathcal{O}})$  and  $\phi(\lambda(t)) = q^{-\langle \lambda, \check{\rho} \rangle}$  (here  $\check{\rho} \in \check{\Lambda}$  is half the sum of positive roots of  $G$  and  $\langle, \rangle$  is the canonical pairing of the weight and coweight lattices).

1.1.5. In [9] we considered a map  $\mathbf{F} : \mathbf{H} \rightarrow \mathbf{W}$  defined by the formula  $\mathbf{F}(h) = \phi_0 \star h$ , i.e.,

$$(\mathbf{F}(h))(g) = \int_{N(\hat{\mathcal{K}})} h(n^{-1} \cdot g) \chi(n) dn$$

(here the measure on  $N(\hat{\mathcal{K}})$  is chosen in such a way that the measure of  $N(\hat{\mathcal{O}})$  is 1). It is easy to see that  $\mathbf{F}$  is an isomorphism of  $\mathbf{H}$ -modules.

Remarkably, it turns out that  $\mathbf{F}$  is compatible with the above bases:

1.1.6. **Theorem.** ([9])  $\mathbf{F}(A_\lambda) = \phi_\lambda$  for all  $\lambda \in \Lambda^{++}$ .

1.1.7. *Connection with the Whittaker functions.* Each semi-simple conjugacy class  $\gamma$  of the group  ${}^L G(\overline{\mathbb{Q}}_\ell)$  defines a homomorphism  $\gamma : \text{Rep } {}^L G(\overline{\mathbb{Q}}_\ell) \rightarrow \overline{\mathbb{Q}}_\ell$ , which maps  $[V]$  to  $\text{Tr}(\gamma, V)$ . We denote the corresponding homomorphism  $\mathbf{H} \rightarrow \overline{\mathbb{Q}}_\ell$  by the same symbol.

For each  $\gamma$  as above one defines the *Whittaker function*  $W_\gamma$  as the unique function on  $G(\hat{\mathcal{K}})$  which satisfies:

- $W_\gamma(g \cdot x) = W_\gamma(g), \forall x \in G(\hat{\mathcal{O}});$
- $W_\gamma(n \cdot g) = \chi(n) \cdot W_\gamma(g), \forall n \in N(\hat{\mathcal{K}});$
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$$h \star W_\gamma = \gamma(h) \cdot W_\gamma, \quad \forall h \in \mathbf{H}$$

(here  $\star$  is defined by the same formula as in (1.3)).

- $W_\gamma(1) = 1.$

It was proved in [9] that Theorem 1.1.6 is equivalent to the following well-known Casselman–Shalika formula (see [5, 22]):

$$(1.4) \quad W_\gamma = \sum_{\lambda \in \Lambda^{++}} \text{Tr}(\gamma, (V^\lambda)^*) \cdot \phi_\lambda.$$

Thus, the functions  $\phi_\lambda$  can be viewed as the “building blocks” for the Whittaker function.

It is instructive to compare formula (1.4) to the formula for the spherical function given in Sect. 7 of [9]; the building blocks for the latter are the functions  $A_\lambda$ .

1.1.8. Theorem 1.1.6 allows to compute  $\phi_\mu \star A_\lambda$  for an arbitrary  $\mu \in \Lambda^{++}$ , as follows.

By Theorem 1.1.2, the structure constants of the Hecke algebra are equal to those of the Grothendieck ring  $\text{Rep } {}^L G$  (the “Clebsch–Gordan coefficients”). We have:

$$A_\lambda \star A_\mu = \sum_{\nu \in \Lambda^+} C_{\lambda\mu}^\nu A_\nu,$$

where  $C_{\lambda\mu}^\nu = \dim \text{Hom}_{{}^L G}(V^\nu, V^\lambda \otimes V^\mu)$ . Now Theorem 1.1.6 implies that

$$(1.5) \quad \phi_\mu \star A_\lambda = \sum_{\nu \in \Lambda^+} C_{\lambda\mu}^\nu \phi_\nu.$$

For  $\mu \in \Lambda$ , let  $\chi_\mu : N(\hat{\mathcal{K}}) \rightarrow \overline{\mathbb{Q}}_\ell^\times$  be a character defined by the formula

$$\chi_\mu(n) = \chi(\mu(t) \cdot n \cdot \mu(t)^{-1}).$$

By evaluating both sides of (1.5) on  $g = (\lambda + \mu)(t)$  we obtain the following:

$$(1.6) \quad \int_{N(\hat{\mathcal{K}})} A_\lambda(n^{-1} \cdot \nu(t)) \chi_\mu(n) dn = q^{-\langle \nu, \rho \rangle} \cdot C_{\lambda\mu}^{\mu+\nu}.$$

## 1.2. Geometrization.

1.2.1. The goal of the present paper is to provide a geometric interpretation of the results described in Sect. 1.1, in particular, of Theorem 1.1.6 and formula (1.6).

The starting point is the fact that the Hecke algebra  $\mathbf{H}$  admits a natural geometric counterpart.

There exists an algebraic group (resp., an ind-group) over  $\mathbb{F}_q$ , whose set of  $\mathbb{F}_q$ -points identifies with  $G(\hat{\mathcal{O}})$  (resp,  $G(\hat{\mathcal{K}})$ ); to simplify notation, we will denote these objects by the same symbols. The quotient  $\mathrm{Gr} := G(\hat{\mathcal{K}})/G(\hat{\mathcal{O}})$  is an ind-scheme and we can consider the category  $P_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$  of  $G(\hat{\mathcal{O}})$ -equivariant perverse sheaves on  $\mathrm{Gr}$ , defined over the algebraic closure  $\overline{\mathbb{F}}_q$  of  $\mathbb{F}_q$  (since the closures of  $G(\hat{\mathcal{O}})$ -orbits in  $\mathrm{Gr}$  are finite-dimensional, there is no problem to define this category).

The  $\overline{\mathbb{Q}}_\ell$ -vector space  $\mathbf{H}$  is isomorphic to the Grothendieck group of the category  $P_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$ . The irreducible objects of  $P_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$  are the intersection cohomology sheaves  $\mathcal{A}_\lambda$  attached to the corresponding  $G(\hat{\mathcal{O}})$ -orbits  $\mathrm{Gr}^\lambda$ . A remarkable fact is that the class of  $\mathcal{A}_\lambda$  in the Grothendieck group corresponds (up to a sign) to the element  $A_\lambda \in \mathbf{H}$  defined in Sect. 1.1.3.

Moreover, Theorem 1.1.2 has a categorical version (Theorem 5.2.6):  $P_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$  is a tensor category and as such it is equivalent to the category  $\mathcal{R}ep({}^L G)$  of finite-dimensional representations of  ${}^L G$ .

1.2.2. Now we would like to find a geometric counterpart  $P_{N(\hat{\mathcal{K}})}^\chi(\mathrm{Gr})$  of the Whittaker module  $\mathbf{W}$ . A natural candidate for it would be a category of  $(N(\hat{\mathcal{K}}), \chi)$ -equivariant perverse sheaves on  $\mathrm{Gr}$ , whose Grothendieck group would be isomorphic to  $\mathbf{W}$ .

Unfortunately, the situation here is much more complicated, since the orbits of the group  $N(\hat{\mathcal{K}})$  on  $\mathrm{Gr}$  are infinite-dimensional and we do not yet have a satisfactory theory of perverse sheaves with infinite-dimensional supports.

In this paper we propose a substitute for the category of  $(N(\hat{\mathcal{K}}), \chi)$ -equivariant perverse sheaves on  $\mathrm{Gr}$ . Our category will consist of perverse sheaves, which, however, do not “live” on  $\mathrm{Gr}$ , but rather on an algebraic stack  $\overline{\mathrm{Bun}}_N$ , which is defined using a global curve  $X$  (see Sect. 5.4.4).

1.2.3. The stack  $\overline{\mathrm{Bun}}_N$  is the Drinfeld compactification of the moduli space of  $N$ -bundles on  $X$ . This stack has been used in dealing with several other problems of geometric representation theory [8, 7, 4].

Let us sketch the definition in the simplest example of  $G = GL(2)$ . In this case, the stack  $\mathrm{Bun}_N$  classifies short exact sequences

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X \rightarrow 0,$$

where  $\mathcal{O}_X$  is the structure sheaf on  $X$  and  $\mathcal{E}$  is a rank two vector bundle. In other words, it classifies rank two vector bundles  $\mathcal{E}$  with  $\det \mathcal{E} \simeq \mathcal{O}_X$ , together with an embedding of the trivial line bundle into  $\mathcal{E}$  (i.e., a *maximal* embedding of sheaves  $\mathcal{O}_X \hookrightarrow \mathcal{E}$ ).

The “compactification”  $\overline{\mathrm{Bun}}_N$  of  $\mathrm{Bun}_N$  classifies rank two vector bundles  $\mathcal{E}$  (with  $\det \mathcal{E} \simeq \mathcal{O}_X$ ), together with an *arbitrary* embedding of sheaves  $\mathcal{O}_X \hookrightarrow \mathcal{E}$ .

The definition in the general case follows the same lines and we refer the reader to Sect. 2 for details.

Informally, from the point of view of  $P_{N(\hat{\mathcal{K}})}^X(\mathrm{Gr})$ , the idea of introducing  $\overline{\mathrm{Bun}}_N$  is as follows. Since  $N(\hat{\mathcal{K}})$ -orbits on  $\mathrm{Gr}$  are infinite-dimensional, it is natural to try to “quotient them out” by a subgroup of  $N(\hat{\mathcal{K}})$ , which acts almost freely on the  $N(\hat{\mathcal{K}})$ -orbits in  $\mathrm{Gr}$ . There is a natural candidate for such a subgroup: fix a projective curve  $X$  and a point  $x \in X$  (so that  $\hat{\mathcal{K}} \simeq \hat{\mathcal{K}}_x$ ) and consider the group  $N_{\mathrm{out}} \subset N(\hat{\mathcal{K}})$  of maps  $(X - x) \rightarrow N$ .

The quotient  $S/N_{\mathrm{out}}$  of a single  $N(\hat{\mathcal{K}})$ -orbit  $S$  on  $\mathrm{Gr}$  makes perfect sense: this is the stack  $\mathrm{Bun}_N$ . In Sect. 7.1.1 we give a precise scheme-theoretic definition of the closure  $\overline{S}$  of  $S$ . This is an ind-scheme, which appears to be highly non-reduced (see Remark 7.1.2). Unlike the case of  $S$ , it is not quite clear what the quotient of  $\overline{S}$  by  $N_{\mathrm{out}}$  should be, since in general the quotient of an ind-scheme by an ind-group is not an algebraic stack.

In fact, there is a natural morphism  $\overline{S} \rightarrow \overline{\mathrm{Bun}}_N$ . The image of the corresponding map at the level of  $\mathbb{F}_q$ -points is the set-theoretic quotient  $\overline{S}/N_{\mathrm{out}}$ . But in general it is a constructible subset of  $\overline{\mathrm{Bun}}_N(\mathbb{F}_q)$ . For example, in the case of  $G = SL_2$ , locally this subset looks like  $\mathbb{A}^2 - (\mathbb{A}^1 - \mathrm{pt})$  inside  $\mathbb{A}^2$ . Thus, there is no obvious way to interpret the naive set-theoretic quotient  $\overline{S}/N_{\mathrm{out}}$  as the set of points of an algebraic variety or an algebraic stack. On the other hand,  $\overline{\mathrm{Bun}}_N$  is a well-defined algebraic stack, and so it may be viewed as an appropriate replacement for  $\overline{S}/N_{\mathrm{out}}$ .

An important feature of  $\overline{\mathrm{Bun}}_N$  is that it has a stratification analogous to the stratification of  $\overline{S}$  by  $N(\hat{\mathcal{K}})$ -orbits (see Corollary 2.2.9). But the dimensions of the strata on  $\overline{\mathrm{Bun}}_N$  are all even and equal to twice the naive (relative) dimensions of the corresponding  $N(\hat{\mathcal{K}})$ -orbits on  $\mathrm{Gr}$ .

1.2.4. The stack  $\overline{\mathrm{Bun}}_N$  (more precisely, its generalization  ${}_{x,\infty}\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$  introduced in Sect. 2.3) allows us to introduce the sought after category  $P_{N(\hat{\mathcal{K}})}^X(\mathrm{Gr})$ . It will possess the following basic properties, motivated by the analogous properties of the Whittaker module  $\mathbf{W}$ :

- (1) The tensor category  $P_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$  acts on  $P_{N(\hat{\mathcal{K}})}^X(\mathrm{Gr})$  by “Hecke convolution functors”.
- (2) The irreducible objects  $\{\overline{\Psi}^\lambda\}$  in  $P_{N(\hat{\mathcal{K}})}^X(\mathrm{Gr})$  are labeled by  $\lambda \in \Lambda^{++}$  and the functor  $\mathcal{F} : P_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr}) \rightarrow P_{N(\hat{\mathcal{K}})}^X(\mathrm{Gr})$  given by  $\mathcal{F}(\mathcal{S}) \rightarrow \overline{\Psi}^0 \star \mathcal{S}$  sends  $\mathcal{A}_\lambda$  to  $\overline{\Psi}^\lambda$ .
- (3) Each  $\overline{\Psi}^\lambda$  (in spite of being an irreducible perverse sheaf) is an extension by 0 of a local system from the corresponding stratum of the stack  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ .

1.2.5. As we will see in Sect. 6.1, the above properties (1) and (2) of  $P_{N(\hat{\mathcal{K}})}^X(\mathrm{Gr})$  formally imply that this category is semi-simple and that the functor  $\mathcal{F}$  is an equivalence of categories. This will in turn imply (almost tautologically) property (3).

Property (3) of the sheaves  $\overline{\Psi}^\lambda$  (which are the sheaf theoretic counterparts of the functions  $\phi_\lambda$ ) is perhaps the most intriguing aspect of the category  $P_{N(\hat{\mathcal{K}})}^X(\mathrm{Gr})$  and is the key point of this paper. We repeat that it means that  $\overline{\Psi}^\lambda$  has zero stalks outside of the

locus where it is a local system. An irreducible perverse sheaf satisfying this property is called “clean” and the appearance of such perverse sheaves in representation theory always has remarkable consequences.

To contrast this to the case of  $G(\hat{\mathcal{O}})$ -equivariant sheaves, note that while the category of  $P_{G(\hat{\mathcal{O}})}(\text{Gr})$  is semi-simple, the irreducible objects  $\mathcal{A}_\lambda$  of  $P_{G(\hat{\mathcal{O}})}(\text{Gr})$  have non-zero stalks on all  $G(\hat{\mathcal{O}})$ -orbits that lie in the closure of the orbit  $\text{Gr}^\lambda$ .

**1.2.6. Remark.** As we have seen above,  $N(\hat{\mathcal{K}})$ -orbits on  $\text{Gr}$  are enumerated by  $\lambda \in \Lambda$  but only those corresponding to  $\lambda \in \Lambda^{++}$  can carry elements of  $\mathbf{W}$ .

However, the stack  ${}_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  will have many more strata, among which we single out what we call the relevant ones (the latter will be in bijection with  $\Lambda^{++}$ ). The perverse sheaves belonging to  $P_{N(\hat{\mathcal{K}})}^X(\text{Gr})$  will have non-zero stalks only on the relevant strata.

**1.2.7.** Another source of motivation for us was the following. Although our definition of  $P_{N(\hat{\mathcal{K}})}^X(\text{Gr})$  is not obvious, it is easy to formulate a sheaf counterpart of formula (1.6).

Denote by  $\overline{\text{Gr}}^\lambda$  the closure of the orbit  $G(\hat{\mathcal{O}}) \cdot \lambda(t)$  in  $\text{Gr}$  and by  $S^\nu$  the  $N(\hat{\mathcal{K}})$ -orbit of  $\nu(t) \in \text{Gr}$ . The variety  $\overline{\text{Gr}}^\lambda \cap S^\nu$  is finite-dimensional and if  $\mu \in \Lambda$  is such that  $\mu + \nu$  is dominant, the character  $\chi_\mu : N(\hat{\mathcal{K}}) \rightarrow \mathbb{G}_a$  gives rise to a map  $\chi_\mu^\nu : \overline{\text{Gr}}^\lambda \cap S^\nu \rightarrow \mathbb{G}_a$  (cf. Sect. 7.1.4 for the definition of  $\chi_\mu^\nu$ ). Let  $\mathcal{I}_\psi$  be the Artin-Schreier sheaf on  $\mathbb{G}_a$ .

The following is a generalization of our Conjecture 7.2. from [9] (in [9] it was stated for  $\mu = 0$ , and in that form it was subsequently proved by B. C. Ngo [19] for  $G = GL_n$ ).

**Theorem 1.** *For  $\lambda \in \Lambda^{++}$  and  $\mu, \nu \in \Lambda$  with  $\mu + \nu \in \Lambda^{++}$  the cohomology*

$$(1.7) \quad H_c^k(\overline{\text{Gr}}^\lambda \cap S^\nu, \mathcal{A}_\lambda|_{\overline{\text{Gr}}^\lambda \cap S^\nu} \otimes \chi_\mu^\nu|_{\overline{\text{Gr}}^\lambda \cap S^\nu}^* (\mathcal{I}_\psi))$$

*vanishes unless  $k = \langle 2\nu, \check{\rho} \rangle$  and  $\mu \in \Lambda^{++}$ . In the latter case, this cohomology identifies canonically with  $\text{Hom}_{L_G}(V^\lambda \otimes V^\mu, V^{\mu+\nu})$ .*

At the end of this paper we will prove this theorem using our category  $P_{N(\hat{\mathcal{K}})}^X(\text{Gr})$  and mainly the “cleanness” property of the objects  $\overline{\Psi}^\lambda$  mentioned above.<sup>2</sup>

It will turn out (as in the function-theoretic setting) that the stalks of the convolution  $\overline{\Psi}^{\mu+\nu} \star \mathcal{A}_{-w_0(\lambda)}$  are isomorphic to the cohomology groups appearing in (1.7).

By passing to the traces of the Frobenius, the above theorem entails formula (1.6), and hence (in the case  $\mu = 0$ ) the Casselman-Shalika formula (1.4), as explained in Sect. 6 of [9]. Thus, we obtain a geometric proof of the Casselman-Shalika formula.

**1.3. Contents.** The main part of the paper is devoted to the study of the Drinfeld compactification  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$  and a certain category of perverse sheaves on it.

We start out in Sect. 2 with the definition of the stack  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$  and its ind-version  ${}_{\overline{x},\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$ .

<sup>2</sup>This approach to the Casselman-Shalika formula was suggested to one of us (D.G.) by his thesis advisor, J. Bernstein, several years ago.

In Sect. 3 we focus on the properties of  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$  that are related to the  $N(\hat{\mathcal{K}})$ -action. An important fact proved in Sect. 3 is that the embedding  $\text{Bun}_N^{\mathcal{F}_T} \hookrightarrow \overline{\text{Bun}}_N^{\mathcal{F}_T}$  is affine, which is an analogue of the fact that the embedding  $S^\nu \hookrightarrow \overline{S}^\nu$  is affine. In addition, we prove that  $\text{Bun}_N^{\mathcal{F}_T}$  is topologically “contractible” (by analogy with  $S^\nu$ , which is isomorphic to an infinite-dimensional affine space).

In Sect. 4 we introduce our main objects of study—the perverse sheaves  $\overline{\Psi}_\varpi$  on  $\overline{\text{Bun}}_N$  or, rather, on its twisted version  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ . We formulate Theorem 2, which expresses the cleanness property of the perverse sheaves  $\overline{\Psi}_\varpi$ , and its generalization, Theorem 3, which is the main result of this paper.

In Sect. 5 we introduce the main tool needed for the proof of Theorem 3 – the action of the category  $\text{P}_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$  on the derived category of sheaves on  ${}_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  by Hecke functors.

In Sect. 6 we derive Theorem 3 from Theorem 4, which describes the action of the Hecke functors on our basic perverse sheaves.

Sect. 7 is devoted to the proof of Theorem 4.

Finally, in Sect. 8 we prove a generalization of the conjecture from [9] (see Theorem 1 above).

**1.4. Notation and conventions.** From now on, we replace the field  $\mathbb{F}_q$  by its algebraic closure  $\overline{\mathbb{F}}_q$ .

Throughout the paper,  $X$  will be a fixed smooth projective connected curve over  $\overline{\mathbb{F}}_q$ .

Furthermore,  $G$  will stand for a connected reductive group over  $\overline{\mathbb{F}}_q$ . We fix a Borel subgroup  $B \subset G$  and let  $N$  be its unipotent radical and  $T = B/N$ . We denote by  $\Lambda$  the *coweight* lattice and by  $\check{\Lambda}$  the *weight* lattice of  $T$ ;  $\langle, \rangle$  denotes the natural pairing between the two.

The set of vertices of the Dynkin diagram of  $G$  is denoted by  $\mathcal{J}$ ;  $\Lambda^{++}$  denotes the semigroup of dominant coweights and  $\Lambda^+$  the span over  $\mathbb{Z}_+$  of positive coroots;  $\check{\Lambda}^{++}$  and  $\check{\Lambda}^+$  denote similar objects for  $\check{\Lambda}$ ;  $\check{\rho} \in \check{\Lambda}$  is the half sum of positive roots of  $G$  and  $w_0$  is the longest element of the Weyl group.

For  $\check{\lambda} \in \check{\Lambda}^{++}$ , we write  $\mathcal{V}^{\check{\lambda}}$  for the corresponding Weyl module, i.e.,

$$\mathcal{V}^{\check{\lambda}} = \Gamma(G/B, \mathcal{L}_{-w_0(\check{\lambda})})^*,$$

where  $\mathcal{L}_{-w_0(\check{\lambda})}$  is the line bundle on  $G/B$  corresponding to the character  $-w_0(\check{\lambda})$  of  $T$ . We obtain a collection of canonical embeddings  $\mathcal{V}^{\check{\lambda}+\check{\mu}} \rightarrow \mathcal{V}^{\check{\lambda}} \otimes \mathcal{V}^{\check{\mu}}$  for all  $\check{\lambda}, \check{\mu} \in \check{\Lambda}^{++}$ .

For  $\lambda \in \Lambda^{++}$ , we denote by  $V^\lambda$  the irreducible representation of  ${}^L G(\overline{\mathbb{Q}}_\ell)$  with highest weight  $\lambda$ .

This paper will extensively use the language of algebraic stacks (over  $\overline{\mathbb{F}}_q$ ), see [16]. When we say that a stack  $\mathcal{Y}$  classifies *something*, it should always be clear what an  $S$ -family of *something* is for any  $\overline{\mathbb{F}}_q$ -scheme  $S$ , i.e., what is the groupoid  $\text{Hom}(S, \mathcal{Y})$ , and what are the functors  $\text{Hom}(S_2, \mathcal{Y}) \rightarrow \text{Hom}(S_1, \mathcal{Y})$  for each morphism  $S_1 \rightarrow S_2$ .

For example, if  $H$  is an algebraic group, we define  $\text{Bun}_H$  as a stack that classifies  $H$ -bundles on  $X$ . This means that  $\text{Hom}(S, \text{Bun}_H)$  is the groupoid whose objects are

$H$ -bundles on  $X \times S$  and morphisms are isomorphisms of these bundles. Pull-back functor for  $S_1 \rightarrow S_2$  is defined in a natural way.

Let  $x \in X$  be a point. We will denote by  $\hat{\mathcal{O}}_x$  (resp.,  $\hat{\mathcal{K}}_x$ ) the completion of  $\mathcal{O}_X$  at  $x$  (resp., the field of fractions of  $\hat{\mathcal{O}}_x$ ). We will use the notation  $\mathcal{D}_x$  (resp.,  $\mathcal{D}_x^\times$ ) for the formal (resp., formal punctured) disc around  $x$  in  $X$ . They will not appear as schemes and we will use them only in the following circumstances:

Let  $S$  be a scheme and assume for simplicity that  $S$  is affine,  $S = \operatorname{Spec}(\mathcal{O}_S)$ . An  $S$ -family of vector bundles on  $\mathcal{D}_x$  (resp.,  $\mathcal{D}_x^\times$ ) is by definition a projective finitely generated module over the completed tensor product  $\mathcal{O}_S \hat{\otimes} \hat{\mathcal{O}}_x$  (resp.,  $\mathcal{O}_S \hat{\otimes} \hat{\mathcal{K}}_x$ ). Similarly, one defines an  $S$ -family of  $G$ -bundles on  $\mathcal{D}_x$  or  $\mathcal{D}_x^\times$ .

Let  $S$  be as above and let  $Y$  be another affine scheme  $Y = \operatorname{Spec}(\mathcal{O}_Y)$ . An  $S$ -family of maps  $\mathcal{D}_x \rightarrow Y$  (resp.,  $\mathcal{D}_x^\times \rightarrow Y$ ) is by definition a ring homomorphism  $\mathcal{O}_Y \rightarrow \mathcal{O}_S \hat{\otimes} \hat{\mathcal{O}}_x$  (resp.,  $\mathcal{O}_Y \rightarrow \mathcal{O}_S \hat{\otimes} \hat{\mathcal{K}}_x$ ).

This paper deals with perverse sheaves on algebraic stacks. Although these objects make a perfect sense, the corresponding derived category is problematic: there is no doubt that the “correct” derived category exists, but a satisfactory definition is still unavailable in the published literature.

For the needs of our paper, we could stay within the abelian category of perverse sheaves, and formulate all of our statements and proofs in terms of complexes of perverse sheaves and their cohomologies. However, doing so would be very inconvenient as it would force us to use cumbersome notation and increase the length of the proofs. For this reason, in this paper we will adopt the point of view that the appropriate definition of the derived category  $\operatorname{Sh}(\mathcal{Y})$  on a stack  $\mathcal{Y}$  exists. The interested reader can easily reformulate all of the statements below so as to avoid the use of the derived category.

Thus, when we discuss objects of the derived category, the cohomological gradation should always be understood in the perverse sense. In addition, for a morphism  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ , the functors  $f_!$ ,  $f_*$ ,  $f^*$  and  $f^!$  should be understood “in the derived sense”.

Let  $\mathcal{Y}$  be a stack,  $H$  an algebraic group and  $\mathcal{F}_H$  an  $H$ -torsor over  $\mathcal{Y}$ . Let, in addition,  $Z$  be an  $H$ -scheme. We denote by  $\mathcal{F}_H \times_H Z$  the corresponding fibration over  $\mathcal{Y}$ , with the typical fiber  $Z$ . (Unfortunately, the notation  $A \times_C B$  is also used for the fiber product of  $A$  and  $B$  over  $C$ , but in all instances it will be clear from the context what the notation means).

Now, if  $\mathcal{T}$  is a perverse sheaf (or a complex) on  $\mathcal{Y}$  and  $\mathcal{S}$  is an  $H$ -equivariant perverse sheaf (or complex) on  $Z$ , we can form their twisted external product and obtain a perverse sheaf (resp., a complex) on  $\mathcal{F}_H \times_H Z$ , denoted  $\mathcal{T} \tilde{\boxtimes} \mathcal{S}$ , which is  $\mathcal{T}$  “along the base” and  $\mathcal{S}$  “along the fiber”.

## 2. DRINFELD’S COMPACTIFICATION

From now on, with the exception of Sect. 8, we will work under the assumption that the derived group  $[G, G]$  is *simply-connected*.

### 2.1. The stack $\operatorname{Bun}_N$ .

2.1.1. Consider the moduli stack  $\mathrm{Bun}_G$  of principal  $G$ -bundles over the curve  $X$ . We repeat that by definition, for an  $\overline{\mathbb{F}}_q$ -scheme  $S$ ,  $\mathrm{Hom}(S, \mathrm{Bun}_G)$  is the groupoid, whose objects are  $G$ -bundles on  $X \times S$  and morphisms are isomorphisms of these bundles. One defines similarly the stacks  $\mathrm{Bun}_B$ ,  $\mathrm{Bun}_T$  and  $\mathrm{Bun}_N$ .

We have natural morphisms of stacks

$$\mathfrak{p} : \mathrm{Bun}_B \rightarrow \mathrm{Bun}_G, \quad \mathfrak{q} : \mathrm{Bun}_B \rightarrow \mathrm{Bun}_T,$$

that send a given  $B$ -bundle  $\mathcal{F}_B$  to the bundles

$$\mathfrak{p}(\mathcal{F}_B) = \mathcal{F}_G := \mathcal{F}_B \times_B G, \quad \mathfrak{q}(\mathcal{F}_B) = \mathcal{F}_T := \mathcal{F}_B \times_B T = \mathcal{F}_B/N,$$

respectively.

The projection  $\mathfrak{p}$  is representable, but in general is not smooth and has non-compact fibers, while the projection  $\mathfrak{q}$  is smooth, but in general non-representable.

Denote by  $\mathcal{F}_T^0$  the trivial  $T$ -bundle on  $X$ . It follows from the definitions that the closed substack  $\mathfrak{q}^{-1}(\mathcal{F}_T^0) \subset \mathrm{Bun}_B$  can be identified with the stack  $\mathrm{Bun}_N$ . In what follows, for a fixed  $T$ -bundle  $\mathcal{F}_T$  we will denote by  $\mathrm{Bun}_N^{\mathcal{F}_T}$  the closed substack  $\mathfrak{q}^{-1}(\mathcal{F}_T) \subset \mathrm{Bun}_B$ .

Here is a Plücker type description of the stack  $\mathrm{Bun}_N^{\mathcal{F}_T}$ :

2.1.2. Given a scheme  $S$ ,  $\mathrm{Hom}(S, \mathrm{Bun}_N^{\mathcal{F}_T})$  is the groupoid whose objects are pairs  $(\mathcal{F}_G, \kappa)$ , where  $\mathcal{F}_G$  is a  $G$ -bundle on  $X \times S$  and  $\kappa = \{\kappa^{\check{\lambda}}\}$  is a collection of *maximal* embeddings

$$(2.1) \quad \kappa^{\check{\lambda}} : \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}, \quad \forall \check{\lambda} \in \check{\Lambda}^{++}.$$

Here  $\mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}}$  is the line bundle on  $X \times S$  induced from  $\mathcal{F}_T$  by means of the character  $\check{\lambda} : T \rightarrow \mathbb{G}_m$ ,  $\mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$  is the vector bundle on  $X \times S$  associated with the representation  $\mathcal{V}^{\check{\lambda}}$  of  $G$  and the  $G$ -bundle  $\mathcal{F}_G$ .

Recall that an embedding of locally free sheaves is called maximal if it is a bundle map, i.e., an injective map of coherent sheaves, such that the quotient is torsion-free.

The above collection  $\kappa$  of embeddings must satisfy the so-called Plücker relations. Namely, given a pair of dominant integral weights of  $G$ ,  $\check{\lambda}$  and  $\check{\mu}$ , the map

$$\mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \otimes \mathcal{L}_{\mathcal{F}_T}^{\check{\mu}} \xrightarrow{\kappa^{\check{\lambda}} \otimes \kappa^{\check{\mu}}} \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}} \otimes \mathcal{V}_{\mathcal{F}_G}^{\check{\mu}} \simeq (\mathcal{V}^{\check{\lambda}} \otimes \mathcal{V}^{\check{\mu}})_{\mathcal{F}_G}$$

must coincide with the composition

$$\mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \otimes \mathcal{L}_{\mathcal{F}_T}^{\check{\mu}} \simeq \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda} + \check{\mu}} \xrightarrow{\kappa^{\check{\lambda} + \check{\mu}}} \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda} + \check{\mu}} \hookrightarrow (\mathcal{V}^{\check{\lambda}} \otimes \mathcal{V}^{\check{\mu}})_{\mathcal{F}_G},$$

where the last arrow comes from the canonical embedding of representations  $\mathcal{V}^{\check{\lambda} + \check{\mu}} \rightarrow \mathcal{V}^{\check{\lambda}} \otimes \mathcal{V}^{\check{\mu}}$  (see Sect. 1.4).

The morphisms between two objects  $(\mathcal{F}_G, \kappa)$  and  $(\mathcal{F}'_G, \kappa')$  in  $\mathrm{Hom}(S, \mathrm{Bun}_N^{\mathcal{F}_T})$  are isomorphisms between  $\mathcal{F}_G$  and  $\mathcal{F}'_G$ , which render the corresponding diagrams involving the maps  $\kappa^{\check{\lambda}}, \kappa'^{\check{\lambda}}$  commutative.

The definition of the functor  $\mathrm{Hom}(S_2, \mathrm{Bun}_N^{\mathcal{F}_T}) \rightarrow \mathrm{Hom}(S_1, \mathrm{Bun}_N^{\mathcal{F}_T})$  associated to a morphism  $S_1 \rightarrow S_2$  is straightforward.

2.1.3. *Example of  $G = GL_2$ .* Given a line bundle  $\mathcal{L}$  on  $X$ , let us denote by  $\mathcal{E}_{\mathcal{L}}$  the algebraic stack classifying the short exact sequences:

$$(2.2) \quad 0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X \rightarrow 0.$$

More precisely, the objects of the groupoid  $\text{Hom}(S, \mathcal{E}_{\mathcal{L}})$  are coherent sheaves  $\mathcal{E}$  on  $X \times S$  together with a short exact sequence

$$0 \rightarrow \mathcal{L} \boxtimes \mathcal{O}_S \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X \boxtimes \mathcal{O}_S \rightarrow 0,$$

and morphisms are morphisms between such exact sequences which are identities at the ends.

There exists a canonical map from  $\mathcal{E}_{\mathcal{L}}$  to the affine space  $H^1(X, \mathcal{L})$  which associates to every short exact sequence as above its extension class. Moreover, it is easy to see that the stack  $\mathcal{E}_{\mathcal{L}}$  is isomorphic to the quotient of  $H^1(X, \mathcal{L})$  by the trivial action of the additive group  $H^0(X, \mathcal{L})$ .

Now, for  $G = GL_2$ , a  $T$ -bundle  $\mathcal{F}_T$  is the same as a pair of line bundles  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . Set  $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ . According to the definition, the stack  $\text{Bun}_N^{\mathcal{F}_T}$  is naturally isomorphic to  $\mathcal{E}_{\mathcal{L}}$ .

## 2.2. Compactification.

2.2.1. As we remarked above, the fibers of the projection  $\mathfrak{p} : \text{Bun}_B \rightarrow \text{Bun}_G$  are non-compact. The reason is that the variety of line subbundles of a fixed degree in a given vector bundle is non-compact. However, the latter has a natural compactification: the variety of invertible subsheaves of a fixed degree in a vector bundle, considered as a locally free sheaf. Following this example, V. Drinfeld proposed the following (partial) compactification of the stack  $\text{Bun}_N^{\mathcal{F}_T}$  along the fibers of the projection  $\mathfrak{p}$ .

2.2.2. **Definition.** The stack  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$  classifies pairs  $(\mathcal{F}_G, \kappa)$ , where  $\mathcal{F}_G$  is a  $G$ -bundle on  $X$ , and  $\kappa = \{\kappa^{\check{\lambda}}\}$  is the collection of maps

$$\kappa^{\check{\lambda}} : \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}},$$

which are now embeddings of coherent sheaves (i.e., we do not require any more that the quotient is torsion-free). These embeddings must satisfy the Plücker relations in the same sense as in Definition 2.1.2.

More precisely, the objects of the groupoid  $\text{Hom}(S, \overline{\text{Bun}}_N^{\mathcal{F}_T})$  are pairs  $(\mathcal{F}_G, \kappa)$ , where  $\mathcal{F}_G$  is a  $G$ -bundle on  $X \times S$  and  $\kappa = \{\kappa^{\check{\lambda}}\}$  is a collection of embeddings

$$(2.3) \quad \kappa^{\check{\lambda}} : \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}, \quad \forall \check{\lambda} \in \check{\Lambda}^{++},$$

such that the quotient  $\mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}} / \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}}$  is  $S$ -flat and the Plücker relations hold. Morphisms in  $\text{Hom}(S, \overline{\text{Bun}}_N^{\mathcal{F}_T})$  are defined in the same way as in  $\text{Hom}(S, \text{Bun}_N^{\mathcal{F}_T})$ .

2.2.3. It is clear that  $\mathrm{Bun}_N^{\mathcal{F}_T}$  is an open substack of  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ , and we denote by  $j$  the corresponding open embedding. One can show that  $\mathrm{Bun}_N^{\mathcal{F}_T}$  is dense in  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ . We will not need this fact and the reader is referred to [3] for the proof.

We will denote by  $\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_N^{\mathcal{F}_T} \rightarrow \mathrm{Bun}_G$  the natural projection:

$$\overline{\mathfrak{p}}(\mathcal{F}_G, \kappa) = \mathcal{F}_G.$$

It is clear that the morphism  $\overline{\mathfrak{p}}$  extends  $\mathfrak{p}$ , i.e., that  $\mathfrak{p} = \overline{\mathfrak{p}} \circ j$ . Moreover, we have the following statement, which follows from the fact that the variety of invertible subsheaves of a fixed degree in a given vector bundle is a complete variety.

Consider the natural  $T$ -action on  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ : a point  $\tau \in T$  acts on a point  $(\mathcal{F}_G, \kappa) \in \overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$  by multiplying each  $\kappa^\lambda$  by  $\lambda(\tau)$ . The projection  $\overline{\mathfrak{p}}$  clearly factors as  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T} \rightarrow \overline{\mathrm{Bun}}_N^{\mathcal{F}_T}/T \rightarrow \mathrm{Bun}_G$ .

2.2.4. **Lemma.** *The morphism  $\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_N^{\mathcal{F}_T}/T \rightarrow \mathrm{Bun}_G$  is representable and proper.*

This is the reason why  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$  is called a “compactification” of  $\mathrm{Bun}_N^{\mathcal{F}_T}$ .

2.2.5. *Example.* Let us again consider the case of  $G = GL_2$  using the notation of Sect. 2.1.3.

In this case  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$  is the stack that classifies the triples  $(\mathcal{E}, \kappa_1, \kappa_2)$ , where  $\mathcal{E}$  is a rank 2 bundle on  $X$ ,  $\kappa_1$  (resp.,  $\kappa_2$ ) is a non-zero map  $\mathcal{L}_1 \rightarrow \mathcal{E}$  (resp.,  $\mathcal{E} \rightarrow \mathcal{L}_2$ ) such that the composition  $\kappa_2 \circ \kappa_1$  vanishes and the induced map

$$\det(\mathcal{E}) \rightarrow \mathcal{L}_1 \otimes \mathcal{L}_2$$

is an isomorphism.

The fiber of  $\overline{\mathfrak{p}}$  over a rank 2 bundle  $\mathcal{E}$  is the vector space  $\mathrm{Hom}(\mathcal{L}_1, \mathcal{E})$  with the origin removed.

Note that  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$  is stratified by locally closed substacks  $_d\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ ,  $d \geq 0$ , which classify those triples  $(\mathcal{E}, \kappa_1, \kappa_2)$ , for which the divisor of zeros of  $\kappa_1$  is of order  $d$ . In particular,  $_0\overline{\mathrm{Bun}}_N^{\mathcal{F}_T} = \mathrm{Bun}_N^{\mathcal{F}_T}$ .

Let us describe the  $\overline{\mathbb{F}}_q$ -points of the substacks  $_d\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ . A point of  $\mathrm{Bun}_N^{\mathcal{F}_T}$  is an isomorphism class of extensions

$$0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{L}_2 \rightarrow 0.$$

Hence the set of  $\overline{\mathbb{F}}_q$ -points of  $\mathrm{Bun}_N^{\mathcal{F}_T}$  is  $H^1(X, \mathcal{L})$ , where  $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ . A point of  $_1\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$  is an isomorphism class of extensions

$$0 \rightarrow \mathcal{L}_1(x) \rightarrow \mathcal{E} \rightarrow \mathcal{L}_2(-x) \rightarrow 0,$$

with some fixed  $x \in X$ . Therefore the set of  $\overline{\mathbb{F}}_q$ -points of  $_1\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$  is the set of points of a vector bundle over  $X$  with fiber  $H^1(X, \mathcal{L}(2x))$  over  $x \in X$ . Likewise, we find that the set of  $\overline{\mathbb{F}}_q$ -points of  $_d\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$  is in one-to-one correspondence with the set of points of a vector bundle over  $S^d X$  with the fiber  $H^1(X, \mathcal{L}(2x_1 + \dots + 2x_d))$  over the point  $x_1 + \dots + x_d \in S^d X$ .

2.2.6. The following statement shows that we have a similar stratification of  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$  for general  $G$ . It is here that the assumption that  $[G, G]$  is simply-connected, which was made at the beginning of this section, becomes essential.

Suppose that  $\mathcal{F}'_T$  is a  $T$ -bundle, such that

$$\mathcal{L}_{\mathcal{F}'_T}^{\check{\lambda}} = \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \left( \sum_k \langle \nu_k, \check{\lambda} \rangle \cdot x_k \right), \quad \forall \check{\lambda} \in \check{\Lambda}.$$

Then we will write:

$$\mathcal{F}'_T = \mathcal{F}_T \left( \sum_k \nu_k \cdot x_k \right).$$

2.2.7. **Proposition.** *Let  $(\mathcal{F}_G, \kappa)$  be an  $\overline{\mathbb{F}}_q$ -point of  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ . Then there exists a unique divisor  $D = \sum_k \nu_k \cdot x_k$  with  $\nu_k \in \Lambda^+$  such that for  $\mathcal{F}'_T = \mathcal{F}_T(D)$  the meromorphic maps*

$$\mathcal{L}_{\mathcal{F}'_T}^{\check{\lambda}} \simeq \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \left( \sum_k \langle \nu_k, \check{\lambda} \rangle \cdot x_k \right) \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$$

*are regular everywhere and maximal.*

2.2.8. *Proof.* For each  $i \in \mathcal{J}$ , pick a fundamental weight  $\check{\omega}_i \in \check{\Lambda}^{++}$ .

Let  $D_i$  be the divisor of zeros of the map

$$\kappa^{\check{\omega}_i} : \mathcal{L}_{\mathcal{F}_T}^{\check{\omega}_i} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\omega}_i}.$$

Set  $D = \sum_{i \in \mathcal{J}} D_i \cdot \alpha_i$ . It is easy to check that  $D$  satisfies all the requirements.

2.2.9. **Corollary.**  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$  is stratified by locally closed substacks  ${}_{\gamma} \text{Bun}_N^{\mathcal{F}_T}$ , with  $\gamma = -\sum_{i \in \mathcal{J}} d_i \alpha_i \in -\Lambda^+$ . The substack  ${}_{\gamma} \text{Bun}_N^{\mathcal{F}_T}$  is a bundle over  $\prod_{i \in \mathcal{J}} S^{d_i} X$  whose fiber  ${}_{\overline{x}, \gamma} \text{Bun}_N^{\mathcal{F}_T}$  at

$$\overline{x} = (x_{i,1} + \dots + x_{i,d_i})_{i \in \mathcal{J}}$$

is isomorphic to  $\text{Bun}_N^{\mathcal{F}'_T}$ , where

$$\mathcal{F}'_T = \mathcal{F}_T \left( \sum_{i \in \mathcal{J}} \sum_{m_i=1}^{d_i} \alpha_i \cdot x_{i,m_i} \right).$$

The above substacks  ${}_{\overline{x}, \gamma} \text{Bun}_N^{\mathcal{F}_T}$  give us a stratification of  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ . We note that the codimension of the stratum  ${}_{\overline{x}, \gamma} \text{Bun}_N^{\mathcal{F}_T}$  in  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$  equals  $2 \sum_{i \in \mathcal{J}} d_i$ ; thus, it is always even.

2.2.10. *Remark.* Consider the stack  $\overline{\text{Bun}}_N^{\mathcal{F}_T}/T$  for  $G = GL(2)$ . In [2] and [24] it was shown that an appropriate semi-stability condition defines in  $\overline{\text{Bun}}_N^{\mathcal{F}_T}/T$  an open substack, which is in fact an algebraic variety, and which can be constructed by a series of blow-ups and blow-downs in the projective space  $\mathbb{P}H^1(X, \mathcal{L})$ .

Moreover, this description (referred to as “geometric approximation”) defines on this open substack of  $\overline{\text{Bun}}_N^{\mathcal{F}_T}/T$  a stratification that coincides with that of Corollary 2.2.9.

For general  $G$ , such a description of a semi-stable part of  $\overline{\text{Bun}}_N^{\mathcal{F}_T}/T$  appears to be unknown, and it would be interesting to find one.

### 2.3. Generalizations.

2.3.1. In addition to the stack  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ , it will be convenient for us to consider its generalization described below.

Let  $\bar{x} = \{x_1, \dots, x_n\}$  be a collection of  $n$  distinct points on  $X$  and let  $\bar{\nu}$  be an  $n$ -tuple of elements of  $\Lambda$ . Let  $_{\bar{x}, \bar{\nu}}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  be the stack classifying the data of  $(\mathcal{F}_G, \kappa)$ , where  $\kappa = \{\kappa^{\check{\lambda}}\}$  is now a collection of arbitrary non-zero maps of coherent sheaves

$$\kappa^{\check{\lambda}} : \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}} \left( \sum_k \langle \nu_k, \check{\lambda} \rangle \cdot x_k \right),$$

subject to the Plücker relations.

In particular, when  $n = 1$ ,  $\bar{x} = \{x\}$ ,  $\bar{\nu} = \{\nu\}$ , we will write  $_{x, \nu}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  instead of  $_{\bar{x}, \bar{\nu}}\overline{\text{Bun}}_N^{\mathcal{F}_T}$ .

By definition, the  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$  is nothing but  $_{\bar{x}, \bar{0}}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  and for a general  $\bar{\nu}$  we have a natural isomorphism

$$(2.4) \quad _{\bar{x}, \bar{\nu}}\overline{\text{Bun}}_N^{\mathcal{F}_T} \simeq \overline{\text{Bun}}_N^{\mathcal{F}'_T},$$

where

$$\mathcal{F}'_T = \mathcal{F}_T \left( - \sum_k \nu_k \cdot x_k \right).$$

We will use the notation  $\bar{\nu}' \geq \bar{\nu}$  if for every  $k = 1, \dots, n$ ,  $\nu'_k - \nu_k \in \Lambda^+$ . It is clear that if  $\bar{\nu}' \geq \bar{\nu}$ , then there is a natural closed embedding

$$_{\bar{x}, \bar{\nu}}\overline{\text{Bun}}_N^{\mathcal{F}_T} \hookrightarrow _{\bar{x}, \bar{\nu}'}\overline{\text{Bun}}_N^{\mathcal{F}_T}.$$

We define the ind-stack  $_{\bar{x}, \infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  to be the inductive limit of  $_{\bar{x}, \bar{\nu}}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  as  $\bar{\nu} \in (\Lambda^+)^n$ .

Note that for  $\bar{\nu}, \bar{\nu}' \in \Lambda^n$ , the closed substacks  $_{\bar{x}, \bar{\nu}}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  and  $_{\bar{x}, \bar{\nu}'}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  belong to the same connected component of  $_{\bar{x}, \infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  only if for every  $k$  the projection of  $\nu'_k - \nu_k$  to  $\Lambda / \text{Span}\{\Lambda^+\} \simeq \pi_1(G)$  equals 0.

In what follows, we will continue to denote by  $\bar{\mathbf{p}}$  the natural map  $_{x, \infty}\overline{\text{Bun}}_N^{\mathcal{F}_T} \rightarrow \text{Bun}_G$ .

2.3.2. We will also need the following locally closed substacks of  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ .

Let us denote by

$$j_{\overline{\nu}} : \overline{\text{Bun}}_N^{\mathcal{F}_T} \hookrightarrow \overline{\text{Bun}}_N^{\mathcal{F}_T}$$

the locally closed substack corresponding to those pairs  $(\mathcal{F}_G, \kappa)$ , for which each map

$$\kappa^{\check{\lambda}} : \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}} \left( \sum_k \langle \nu_k, \check{\lambda} \rangle \cdot x_k \right)$$

is regular and maximal for all  $\check{\lambda} \in \check{\Lambda}^{++}$  on the whole of  $X$ .

Note that under the isomorphism (2.4),  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$  goes to  $\text{Bun}_N^{\mathcal{F}_T}$ .

Let us also introduce the locally-closed substack

$$\tilde{j}_{\overline{\nu}} : \widetilde{\overline{\text{Bun}}_N^{\mathcal{F}_T}} \hookrightarrow \overline{\text{Bun}}_N^{\mathcal{F}_T}$$

as the locus corresponding to pairs  $(\mathcal{F}_G, \kappa^{\check{\lambda}})$  for which each map

$$\kappa^{\check{\lambda}} : \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}} \left( \sum_k \langle \nu_k, \check{\lambda} \rangle \cdot x_k \right)$$

is regular and is moreover maximal *in a neighborhood of*  $\bigcup_{k=1}^n x_k$  for all  $\check{\lambda} \in \check{\Lambda}^{++}$ . Note

that this is equivalent to saying that the quotient  $\mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}} \left( \sum_k \langle \nu_k, \check{\lambda} \rangle \cdot x_k \right) / \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}}$  has no torsion supported at any of the points  $x_1, \dots, x_n$ .

To summarize, for each  $\overline{\nu}$  we have a sequence of embeddings:

$$\overline{\text{Bun}}_N^{\mathcal{F}_T} \hookrightarrow \widetilde{\overline{\text{Bun}}_N^{\mathcal{F}_T}} \hookrightarrow \overline{\text{Bun}}_N^{\mathcal{F}_T} \hookrightarrow \overline{\text{Bun}}_N^{\mathcal{F}_T},$$

where the first two arrows are open embeddings and the last arrow is a closed one.

### 3. PROPERTIES OF $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ .

In this section we will prove several important technical facts concerning the structure of  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ . These facts will not be used until Sect. 6.

#### 3.1. Presentation as a double quotient.

3.1.1. *The structure of  $N(\hat{\mathcal{K}}_y)$ .* Let  $y$  be an  $\overline{\mathbb{F}}_q$ -point of  $X$ . Let  $G(\hat{\mathcal{O}}_y)$  (resp.,  $G(\hat{\mathcal{K}}_y)$ ) be the group scheme (resp., group ind-scheme) that classifies maps  $\mathcal{D}_y \rightarrow G$  (resp.,  $\mathcal{D}_y^\times \rightarrow G$ ) (cf. Sect. 1.4). In a similar way we define the corresponding objects for  $N$ , namely  $N(\hat{\mathcal{O}}_y)$  and  $N(\hat{\mathcal{K}}_y)$ .

We will now recall some useful facts about the structure of  $N(\hat{\mathcal{K}}_y)$ . For the most part, the discussion below applies to any unipotent algebraic group (in our case it will be  $N$ ).

Note first that  $N(\hat{\mathcal{K}}_y)$  is not only a group ind-scheme (i.e., a group-like object in the category of ind-schemes), but actually an ind-group scheme. In other words,  $N(\hat{\mathcal{K}}_y)$  can be represented as a direct limit of certain group schemes  $N^{-k}$ ,  $k > 0$ . Moreover, each  $N^{-k}$  is an inverse limit of finite-dimensional unipotent groups.

Furthermore, for each  $k$  we may find a normal subgroup  $N^k$  of  $N^{-k}$ , such that  $N^k \subset N(\hat{\mathcal{O}}_y)$ , and  $N(\hat{\mathcal{O}}_y) = \varprojlim N(\hat{\mathcal{O}}_y)/N^k$ . We denote by  $N_k$  (resp.,  $N'_k$ ) the quotient  $N^{-k}/N^k$  (resp.,  $N(\hat{\mathcal{O}}_y)/N^k$ ).

For example, in the case of  $GL_2$ , when  $N = \mathbb{G}_a$ ,  $N^k$  (resp.,  $N^{-k}$ ) can be chosen to be  $t^{-k}\hat{\mathcal{O}}_y$  (resp.,  $t^k\hat{\mathcal{O}}_y$ ), where  $t$  is a formal coordinate at  $y$ .

3.1.2. So far, our discussion has been local, i.e., we used only the formal neighborhood of  $y$  in  $X$ . Now the curve  $X$  defines a group ind-subscheme  $N_{\text{out}}$  of  $N(\hat{\mathcal{K}}_y)$ , such that  $\text{Hom}(S, N_{\text{out}}) = \text{Hom}(S \times (X - y), N)$ .

Note that any  $\mathbb{G}_a$ -bundle over an affine scheme is trivial. Hence, any  $N$ -bundle over  $(X \setminus y) \times S$ , where  $S$  is any  $\overline{\mathbb{F}}_q$ -scheme, can be trivialized locally on  $S$  in Zariski topology. This allows us to obtain all  $N$ -bundles by “gluing” together trivial bundles over  $X \setminus y$  and  $\mathcal{D}_y$  by a “transition function” on  $\mathcal{D}_y^\times$ , which is an element of  $N(\hat{\mathcal{K}}_y)$ . Thus, informally one can think of  $\text{Bun}_N$  as the double quotient  $N_{\text{out}} \backslash N(\hat{\mathcal{K}}_y) / N(\hat{\mathcal{O}}_y)$ . To avoid quotienting out by the ind-group  $N_{\text{out}}$ , we replace  $N(\hat{\mathcal{K}}_y)$  by a large enough subgroup  $N^{-k}$ . Then, for large enough  $k$ , we obtain a description of  $\text{Bun}_N$  as the double quotient  $N_{\text{out},k} \backslash N^{-k} / N(\hat{\mathcal{O}}_y)$ , where  $N_{\text{out},k} = N_{\text{out}} \cap N^{-k}$ .

3.1.3. The analogous description of  $\text{Bun}_N^{\mathcal{F}_T}$  for any  $T$ -bundle  $\mathcal{F}_T$  over  $X$  is obtained as follows. Let us choose an embedding  $T \xrightarrow{f} B$  and consider the induced  $B$ -bundle  $\mathcal{F}_B^f := \mathcal{F}_T \times^T B$ . Let  $\mathcal{N}^{\mathcal{F}_T}$  be the group scheme of endomorphisms of  $\mathcal{F}_B^f$  that preserve the identification  $\mathcal{F}_B^f / N \simeq \mathcal{F}_T$ .

We have the ind-group scheme  $N^{\mathcal{F}_T}(\hat{\mathcal{K}}_y) = \Gamma(\mathcal{D}_y^\times, \mathcal{N}^{\mathcal{F}_T})$  and its subgroups  $N^{\mathcal{F}_T}(\hat{\mathcal{O}}_y) = \Gamma(\mathcal{D}_y, \mathcal{N}^{\mathcal{F}_T})$  and  $N_{\text{out}}^{\mathcal{F}_T} = \Gamma(X - y, \mathcal{N}^{\mathcal{F}_T})$ .

Let us fix a trivialization of the restriction of a  $T$ -bundle  $\mathcal{F}_T$  to  $\mathcal{D}_y$ :

$$\epsilon : \mathcal{F}_T^0|_{\mathcal{D}_y} \rightarrow \mathcal{F}_T|_{\mathcal{D}_y}.$$

Then we can identify  $N^{\mathcal{F}_T}(\hat{\mathcal{K}}_y)$  and  $N^{\mathcal{F}_T}(\hat{\mathcal{O}}_y)$  with  $N(\hat{\mathcal{K}}_y)$  and  $N(\hat{\mathcal{O}}_y)$ , respectively, and consider  $N_{\text{out}}^{\mathcal{F}_T}$  as a subgroup of  $N(\hat{\mathcal{K}}_y)$ .

Set

$$N_{\text{out},k}^{\mathcal{F}_T} = N_{\text{out}}^{\mathcal{F}_T} \cap N^{-k} \subset N(\hat{\mathcal{K}}_y).$$

This is a finite-dimensional unipotent group scheme for any  $k$  and we have a map  $N_{\text{out},k}^{\mathcal{F}_T} \rightarrow N_k = N^{-k}/N^k$ , which is injective when  $k$  is large enough. Moreover, we have

3.1.4. **Lemma.** *For any  $k$ , a normal subgroup of finite codimension in  $N^k$  acts freely on the quotient  $N_{\text{out},k}^{\mathcal{F}_T} \backslash N^{-k}$ , so the double quotient  $N_{\text{out},k}^{\mathcal{F}_T} \backslash N^{-k} / N(\hat{\mathcal{O}}_y)$  is a well-defined algebraic stack (of finite type). Moreover, we have a natural map*

$$N_{\text{out},k}^{\mathcal{F}_T} \backslash N^{-k} / N(\hat{\mathcal{O}}_y) \rightarrow \text{Bun}_N^{\mathcal{F}_T},$$

which is an isomorphism for  $k$  large enough.

## 3.2. The canonical $N(\hat{\mathcal{O}}_y)$ -torsor.

3.2.1. Let  $\mathcal{F}_G$  be an  $S$ -family of  $G$ -bundles on  $X$ . By making a restriction from  $X$  to  $\mathcal{D}_y$ , we obtain an  $S$ -family of  $G$ -bundles on  $\mathcal{D}_y$  (cf. Sect. 1.4).

Note that an  $S$ -family of  $G$ -bundles on  $\mathcal{D}_y$  is the same as an  $G(\hat{\mathcal{O}}_y)$ -bundle over  $S$ . This construction yields a canonical  $G(\hat{\mathcal{O}}_y)$ -torsor over  $\text{Bun}_G$ , which we will denote by  $\mathcal{G}_y$ .

Analogously, we define the canonical  $B(\hat{\mathcal{O}}_y)$ -bundle  $\mathcal{B}_y$  over  $\text{Bun}_B$ . The datum of  $\epsilon$  gives us a reduction of  $\mathcal{B}_y|_{\text{Bun}_N^{\mathcal{F}_T}}$  to  $N(\hat{\mathcal{O}}_y)$ . We denote the corresponding  $N(\hat{\mathcal{O}}_y)$ -bundle over  $\text{Bun}_N^{\mathcal{F}_T}$  by  $\mathcal{N}_y^\epsilon$ . The important fact is that *the action of  $N(\hat{\mathcal{O}}_y)$  on  $\mathcal{N}_y^\epsilon$  extends naturally to an  $N(\hat{\mathcal{K}}_y)$ -action*. Loosely speaking,  $N(\hat{\mathcal{K}}_y)$  acts by changing the transition function of an  $N$ -bundle on  $\mathcal{D}_y^\times$ . A precise construction of this action is given in the proof of Lemma 3.2.7 below.

3.2.2. Informally, one can say that  $\mathcal{N}_y^\epsilon \simeq N_{\text{out}}^{\mathcal{F}_T} \backslash N(\hat{\mathcal{K}}_y)$ . More precisely, Lemma 3.1.4 implies the following result:

**Corollary.** *For every  $k$  we have a natural map  $N_{\text{out},k}^{\mathcal{F}_T} \backslash N^{-k} \rightarrow \mathcal{N}_y^\epsilon$ , which is an isomorphism for  $k$  large enough.*

Clearly, the  $N^{-k}$  action on  $N_{\text{out},k}^{\mathcal{F}_T} \backslash N^{-k}$  coincides with the one that comes from the above mentioned  $N(\hat{\mathcal{K}}_y)$ -action on  $\mathcal{N}_y^\epsilon$  and the embedding  $N^{-k} \hookrightarrow N(\hat{\mathcal{K}}_y)$ .

3.2.3. Consider the  $N'_k$ -bundle  ${}^k\mathcal{N}_y^\epsilon = \mathcal{N}_y^\epsilon / N^k$  over  $\text{Bun}_N^{\mathcal{F}_T}$ . Lemma 3.1.4 implies:

3.2.4. **Corollary.** *For  $k$  large enough, we have an isomorphism of  $N'_k$ -torsors over  $\text{Bun}_N^{\mathcal{F}_T}$ :*

$$N_{\text{out},k}^{\mathcal{F}_T} \backslash N_k \rightarrow {}^k\mathcal{N}_y^\epsilon.$$

*In particular, for large  $k$ ,  ${}^k\mathcal{N}_y^\epsilon$  is an affine scheme, which is isomorphic to a tower of affine spaces.*

3.2.5. *Example.* Let us illustrate the above discussion on the example of  $G = GL(2)$ . In this case, we have an isomorphism  $\text{Bun}_N^{\mathcal{F}_T} \simeq \mathcal{E}_{\mathcal{L}}$  for an appropriate line bundle  $\mathcal{L}$  (see Sect. 2.1.3). The trivialization  $\epsilon$  identifies  $N(\hat{\mathcal{O}}_y)$  with  $H^0(\mathcal{D}_y, \mathcal{L})$ . The stack  $\text{Bun}_N^{\mathcal{F}_T}$  is isomorphic to

$$H^1(X, \mathcal{L}) / H^0(X, \mathcal{L}) \simeq H^0(X - y, \mathcal{L}) \backslash \hat{\mathcal{K}}_y / \hat{\mathcal{O}}_y.$$

The scheme  $\mathcal{N}_y^\epsilon$  can be identified with

$$H_c^1(X - y, \mathcal{L}) := \varprojlim H^1(X, \mathcal{L}(-k \cdot y)) \simeq H^0(X - y, \mathcal{L}) \backslash \hat{\mathcal{K}}_y.$$

In addition, we have:

$${}^k\mathcal{N}_y^\epsilon \simeq H^1(X, \mathcal{L}(-k \cdot y)) \simeq H^0(X - y, \mathcal{L}) \backslash \hat{\mathcal{K}}_y / H^0(\mathcal{D}_y, \mathcal{L}(-k \cdot y)).$$

3.2.6. Now we would like to extend the  $N(\hat{\mathcal{O}}_y)$ -torsor  $\mathcal{N}_y^\epsilon$  from  $\mathrm{Bun}_N^{\mathcal{F}_T}$  to the compactification  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ . Unfortunately, this does not seem to be possible. The problem is that those points of  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ , for which at least one of the maps

$$\kappa^{\check{\lambda}} : \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$$

has a zero at  $y$ , do not give rise to genuine  $B$ -bundles on  $\mathcal{D}_y$ .

However,  $\mathcal{N}_y^\epsilon$  can be extended to the open substack  ${}_{y,0}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T} \subset \overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ , since by the definition of  ${}_{y,0}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T}$ , the data of  $\epsilon$  gives rise to bundle maps

$$\mathcal{O}|_{\mathcal{D}_y} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}|_{\mathcal{D}_y}, \quad \check{\lambda} \in \check{\Lambda}^{++},$$

which satisfy the Plücker relations. Therefore we obtain a reduction of the  $G(\hat{\mathcal{O}}_y)$ -torsor  $\mathcal{G}_y|_{{}_{y,0}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T}}$  to  $N(\hat{\mathcal{O}}_y)$ .

We denote the resulting  $N(\hat{\mathcal{O}}_y)$ -bundle over  ${}_{y,0}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T}$  by  $\tilde{\mathcal{N}}_y^\epsilon$ . More precisely,  $\tilde{\mathcal{N}}_y^\epsilon$  classifies triples  $(\mathcal{F}_G, \kappa, \varphi)$ , where  $(\mathcal{F}_G, \kappa) \in {}_{y,0}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T}$  and hence gives rise to a well-defined  $N$ -bundle on  $\mathcal{D}_y$ , and  $\varphi$  is a trivialization of this bundle.

The following fact will play an important role in the proof of Theorem 3.

3.2.7. **Lemma.** *The  $N(\hat{\mathcal{O}}_y)$ -action on  $\tilde{\mathcal{N}}_y^\epsilon$  extends naturally to an  $N(\hat{\mathcal{K}}_y)$ -action.*

3.2.8. *Proof.* As was mentioned above, the action will come from “changing the transition functions over  $\mathcal{D}_y^\times$ ”.

For a point<sup>3</sup>  $(\mathcal{F}_G, \kappa, \varphi)$  of  $\tilde{\mathcal{N}}^\epsilon$  and a point  $n \in N(\hat{\mathcal{K}}_y)$ , we will produce a new point  $(\mathcal{F}'_G, \kappa', \varphi')$  of  $\tilde{\mathcal{N}}^\epsilon$  as follows. We set  $(\mathcal{F}'_G, \kappa')|_{X-y} = (\mathcal{F}_G, \kappa)|_{X-y}$ . To define the triple  $(\mathcal{F}'_G, \kappa', \varphi')$  at  $y$ , we fix an affine neighborhood  $X_0$  of  $y$ . The trivialization  $\varphi$  attaches to  $\phi \in H^0(X_0 - y, \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}})$  its Laurent expansion at  $y$ , which we denote by  $\hat{\phi} \in \mathcal{V}^{\check{\lambda}} \otimes \hat{\mathcal{K}}_y$ . We define  $\mathcal{F}'_G$  on  $X_0$  by the following condition: the sections of  $\mathcal{V}_{\mathcal{F}'_G}^{\check{\lambda}}$  on  $X_0$  are precisely the sections  $\phi \in H^0(X_0 - y, \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}})$  such that  $n \cdot \hat{\phi} \in \mathcal{V}^{\check{\lambda}} \otimes \hat{\mathcal{O}}_y$ . It remains to check that the map  $\kappa'$  indeed maps  $\mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}}$  to  $\mathcal{V}_{\mathcal{F}'_G}^{\check{\lambda}}$ . This follows from the fact that the image of the map  $\kappa^{\check{\lambda}}|_{\mathcal{D}_y^\times} : \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}}|_{\mathcal{D}_y^\times} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}|_{\mathcal{D}_y^\times}$  is  $N(\hat{\mathcal{K}}_y)$ -invariant.

Finally, note that the  $G$ -bundle  $\mathcal{F}'_G$  comes with a trivialization over  $\mathcal{D}_y$  and by construction this trivialization is compatible with  $\kappa'$ . This gives the trivialization  $\varphi'$ .

Note that the  $N(\hat{\mathcal{K}}_y)$ -action does not preserve the projection  $\tilde{\mathcal{N}}_y^\epsilon \rightarrow \mathrm{Bun}_N^{\mathcal{F}_T}$ .

3.3. **Affinness.** In this section we prove the following statement, which will be used in the proof of Proposition 6.2.1.

3.3.1. **Proposition.** *The open embedding  ${}_{\bar{x}, \bar{v}}\mathrm{Bun}_N^{\mathcal{F}_T} \hookrightarrow {}_{\bar{x}, \bar{v}}\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$  is affine.*

<sup>3</sup>By a point we mean, as usual, an  $S$ -point for an arbitrary  $\mathbb{F}_q$ -scheme  $S$ .

3.3.2. Recall that we have an isomorphism (2.4), which identifies  $\overline{x, \overline{\nu}} \text{Bun}_N^{\mathcal{F}_T}$  and  $\text{Bun}_N^{\mathcal{F}_T}$  as well as their closures. Therefore to prove Proposition 3.3.1 it suffices to show that the embedding  $\text{Bun}_N^{\mathcal{F}_T} \hookrightarrow \overline{\text{Bun}_N^{\mathcal{F}_T}}$  is affine.

For  $\mu \in \Lambda^+$ , introduce an open substack  $\overline{\text{Bun}_{N, \mu}^{\mathcal{F}_T}}$  of  $\overline{\text{Bun}_N^{\mathcal{F}_T}}$ . It classifies those pairs  $(\mathcal{F}_G, \kappa)$  for which the torsion part of the quotient sheaf  $\mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}} / \text{Im}(\kappa^{\check{\lambda}})$  has length less than or equal to  $\langle \mu, \check{\lambda} \rangle$ , for all  $\check{\lambda} \in \check{\Lambda}^{++}$ .

Clearly,  $\overline{\text{Bun}_{N, 0}^{\mathcal{F}_T}} = \text{Bun}_N^{\mathcal{F}_T}$  and  $\overline{\text{Bun}_{N, \mu}^{\mathcal{F}_T}} \subset \overline{\text{Bun}_{N, \nu}^{\mathcal{F}_T}}$  if  $\nu - \mu \in \Lambda^+$ . For  $y \in X$  let  ${}_{y, 0} \widetilde{\text{Bun}_{N, \mu}^{\mathcal{F}_T}} = {}_{y, 0} \widetilde{\text{Bun}_N^{\mathcal{F}_T}} \cap \overline{\text{Bun}_{N, \mu}^{\mathcal{F}_T}}$ .

It is clear that every  $\overline{\mathbb{F}_q}$ -point of  $\overline{\text{Bun}_N^{\mathcal{F}_T}}$  belongs to  ${}_{y, 0} \widetilde{\text{Bun}_{N, \mu}^{\mathcal{F}_T}}$  for some  $y$  and  $\mu$ . Hence, it is sufficient to show that the embedding of  $\text{Bun}_N^{\mathcal{F}_T}$  into every  ${}_{y, 0} \widetilde{\text{Bun}_{N, \mu}^{\mathcal{F}_T}}$  is affine.

Now Proposition 3.3.1 follows by combining Corollary 3.2.4 and the following result. Denote the  $N'_k$ -bundle  $\widetilde{\mathcal{N}}_y^\epsilon / N_k$  by  ${}^k \widetilde{\mathcal{N}}_y^\epsilon$ .

**3.3.3. Proposition.** *For fixed  $\mathcal{F}_T$ ,  $\mu$  and  $y$ , the restriction of  ${}^k \widetilde{\mathcal{N}}_y^\epsilon$  to  ${}_{y, 0} \widetilde{\text{Bun}_{N, \mu}^{\mathcal{F}_T}}$  is a scheme when  $k$  is large enough.*

3.3.4. *Proof.* First, we claim that the stack  ${}_{y, 0} \widetilde{\text{Bun}_{N, \mu}^{\mathcal{F}_T}}$  is of finite type. This follows from:

**Lemma.** *The image of  ${}_{y, 0} \widetilde{\text{Bun}_{N, \mu}^{\mathcal{F}_T}}$  under the projection  $\overline{\mathfrak{p}} : \overline{\text{Bun}_N^{\mathcal{F}_T}} \rightarrow \text{Bun}_G$  is contained in an open substack  $U$  of  $\text{Bun}_G$  of finite type.*

*Proof of the lemma.* Let  $\check{\lambda} \in \check{\Lambda}^{++}$  be a regular weight and let  $\{\check{\eta}\}$  be the collection of weights of  $\mathcal{V}^{\check{\lambda}}$ . We can choose  $\nu \in \check{\Lambda}$  such that

$$\langle \nu, \check{\lambda} \rangle > \max_{\check{\eta}; 0 \leq \mu' \leq \mu} \langle \mu' + \deg(\mathcal{F}_T), \check{\eta} \rangle.$$

Let  $\mathcal{F}_G$  be a  $G$ -bundle in the image of  ${}_{y, 0} \widetilde{\text{Bun}_{N, \mu}^{\mathcal{F}_T}}$  under the projection  $\overline{\mathfrak{p}} : \overline{\text{Bun}_N^{\mathcal{F}_T}} \rightarrow \text{Bun}_G$ . By definition, it admits a  $B$ -structure of degree  $\deg(\mathcal{F}_T) + \mu'$  for  $0 \leq \mu' \leq \mu$  (here by degree of a  $B$ -bundle we understand the degree of the corresponding  $T$ -bundle, which is an element of  $\Lambda$ ). Then the associated bundle  $\mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$  admits a filtration, whose successive quotients are line bundles of degrees  $\langle \mu' + \deg(\mathcal{F}_T), \check{\eta} \rangle$ .

Therefore  $\mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$  does not admit a line subbundle of degree  $\langle \nu', \check{\lambda} \rangle$  for any  $\nu' \geq \nu$ . Hence,  $\mathcal{F}_G$  does not admit a  $B$ -structure of degree greater than or equal to  $\nu$ . But it is well-known that the open substack of  $\text{Bun}_G$ , which classifies such  $G$ -bundles, is of finite type.  $\square$

Recall the canonical  $G(\hat{\mathcal{O}}_y)$ -bundle  $\mathcal{G}_y$  over  $\text{Bun}_G$  from Sect. 3.2.1. For  $i > 0$ , denote by  $G^i$  the  $i$ -th congruence subgroup of  $G(\hat{\mathcal{O}}_y)$  and by  $G_i$  the quotient  $G(\hat{\mathcal{O}}_y)/G^i$ . The stack-theoretic quotient  ${}^i \mathcal{G}_y := \mathcal{G}_y/G^i$  is a principal  $G_i$ -bundle over  $\text{Bun}_G$ . It is well-known that for any open substack  $U \subset \text{Bun}_G$  of finite type, there exists an integer  $i > 0$  such that the restriction of  ${}^i \mathcal{G}_y$  to  $U$  is a scheme.

Since  $\bar{\mathfrak{p}}$  is representable, we obtain that on the one hand,  ${}^i\mathcal{G}_y|_{\widetilde{{}_{y,0}\mathrm{Bun}_{N,\mu}^{\mathcal{F}_T}}}$  is a scheme. On the other hand, by definition, the induced  $G(\hat{\mathcal{O}}_y)$ -bundle  $\widetilde{\mathcal{N}}_y^\epsilon \times_{N(\hat{\mathcal{O}}_y)} G(\hat{\mathcal{O}}_y)$  can be canonically identified with  $\mathcal{G}_y|_{\widetilde{{}_{y,0}\mathrm{Bun}_{N,\mu}^{\mathcal{F}_T}}}$ .

Now, for every  $i > 0$  we can find an integer  $k > 0$  such that  $N^k \subset N(\hat{\mathcal{O}}_y) \cap G^i$ . Therefore for a given  $i$  and large enough  $k$ , we have an isomorphism of  $G_i$ -bundles:

$${}^k\widetilde{\mathcal{N}}_y^\epsilon \times_{N_k'} G_i \simeq {}^i\mathcal{G}_y|_{\widetilde{{}_{y,0}\mathrm{Bun}_{N,\mu}^{\mathcal{F}_T}}}.$$

Hence for  $i$  and  $k$  large enough,  ${}^k\widetilde{\mathcal{N}}_y^\epsilon \times_{N_k'} G_i$  is a scheme. Since the stack  ${}^k\widetilde{\mathcal{N}}_y^\epsilon$  admits a representable morphism to  ${}^k\widetilde{\mathcal{N}}_y^\epsilon \times_{N_k'} G_i$ , it is also a scheme for  $i$  and  $k$  large enough.

This completes the proof of Proposition 3.3.3 and therefore of Proposition 3.3.1.

#### 4. SHEAVES

##### 4.1. Evaluation morphisms and perverse sheaves.

4.1.1. For a fixed  $\mathcal{F}_T \in \mathrm{Bun}_T$  consider the affine space:  $\prod_{i \in \mathcal{J}} H^1(X, \mathcal{L}_{\mathcal{F}_T}^{\check{\alpha}_i})$ .

We construct a natural morphism:

$$\mathrm{ev} : \mathrm{Bun}_N^{\mathcal{F}_T} \rightarrow \prod_{i \in \mathcal{J}} H^1(X, \mathcal{L}_{\mathcal{F}_T}^{\check{\alpha}_i}).$$

First, let us consider the case  $G = GL_2$ . Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be as in Sect. 2.1.3. Then  $\mathcal{L}_{\mathcal{F}_T}^{\check{\alpha}} \simeq \mathcal{L} := \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ . The map  $\mathrm{ev}$  is just the composition

$$\mathrm{Bun}_N^{\mathcal{F}_T} \simeq \mathcal{E}_{\mathcal{L}} \rightarrow H^1(X, \mathcal{L}).$$

To define the morphism  $\mathrm{ev}$  for general  $G$  we construct for every  $i \in \mathcal{J}$  a two-dimensional representation  $\mathcal{V}^i$  of the group  $B$ . Its restriction to  $T \subset B$  is the direct sum  $\mathbf{1} \oplus \mathbf{1}^{\check{\alpha}_i}$  of the trivial one-dimensional representation  $\mathbf{1}$  and the one-dimensional representation corresponding to the character  $B \rightarrow T \xrightarrow{\check{\alpha}_i} \mathbb{G}_m$ . The subgroup  $N \subset B$  maps  $\mathbf{1}$  to  $\mathbf{1}^{\check{\alpha}_i}$  via the projection

$$N \twoheadrightarrow N/[N, N] \rightarrow \mathrm{Span}\{\check{\alpha}_i\}.$$

Thus, we have a short exact sequence of  $B$ -modules

$$0 \rightarrow \mathbf{1}^{\check{\alpha}_i} \rightarrow \mathcal{V}^i \rightarrow \mathbf{1} \rightarrow 0.$$

By definition, a point  $(\mathcal{F}_G, \kappa) \in \mathrm{Bun}_N^{\mathcal{F}_T}$  defines a  $B$ -bundle  $\mathcal{F}_B$  on  $X$ , such that  $\mathcal{V}_{\mathcal{F}_B}^i$  fits into a short exact sequence:

$$0 \rightarrow \mathcal{L}_{\mathcal{F}_T}^{\check{\alpha}_i} \rightarrow \mathcal{V}_{\mathcal{F}_B}^i \rightarrow \mathcal{O}_X \rightarrow 0.$$

Hence we obtain a morphism of stacks  $\mathrm{Bun}_N^{\mathcal{F}_T} \rightarrow \mathcal{E}_{\mathcal{L}_{\mathcal{F}_T}^{\check{\alpha}_i}}$  (the latter is defined in Sect. 2.1.3). By composing it with the canonical map  $\mathcal{E}_{\mathcal{L}_{\mathcal{F}_T}^{\check{\alpha}_i}} \rightarrow H^1(X, \mathcal{L}_{\mathcal{F}_T}^{\check{\alpha}_i})$  we obtain

a morphism

$$\mathrm{ev}_i : \mathrm{Bun}_N^{\mathcal{F}_T} \rightarrow H^1(X, \mathcal{L}_{\mathcal{F}_T}^{\check{\alpha}_i}).$$

Finally, we set  $\mathrm{ev} = \prod_{i \in \mathcal{I}} \mathrm{ev}_i$ .

4.1.2. To define our sheaves on  $\mathrm{Bun}_N^{\mathcal{F}_T}$  we need to choose additional data.

Suppose that for every  $i \in \mathcal{I}$  we are given an embedding

$$\varpi_i : \mathcal{L}_{\mathcal{F}_T}^{\check{\alpha}_i} \hookrightarrow \Omega.$$

Let  $\varpi$  denote the collection  $\{\varpi_i\}$ ,  $\forall i \in \mathcal{I}$ .

To such  $\varpi$  we assign its conductor,  $\mathrm{cond}(\varpi)$ , which is a divisor on  $X$  with values in the semi-group  $\Lambda_{G_{\mathrm{ad}}}^{++}$  of dominant co-weights of the group  $G_{\mathrm{ad}}$ , where  $G_{\mathrm{ad}} := G/Z(G)$ . Namely, we set  $\langle \mathrm{cond}(\varpi), \check{\alpha}_i \rangle$  equal to the divisor of zeroes of the map  $\varpi_i$  for all  $i \in \mathcal{I}$ .

For a point  $x \in X$ , the conductor of  $\varpi$  at  $x$ , denoted by  $\mathrm{cond}_x(\varpi)$ , is by definition an element of  $\Lambda_{G_{\mathrm{ad}}}^{++}$  equal to the value of  $\mathrm{cond}(\varpi)$  at  $x$ .

4.1.3. Let us fix the data of  $\varpi$ . We define a morphism  $\mathrm{ev}_{\varpi} : \mathrm{Bun}_N^{\mathcal{F}_T} \rightarrow \mathbb{G}_a$  as the composition

$$\mathrm{Bun}_N^{\mathcal{F}_T} \xrightarrow{\mathrm{ev}_i} \prod_{i \in \mathcal{I}} H^1(X, \mathcal{L}_{\mathcal{F}_T}^{\check{\alpha}_i}) \xrightarrow{\varpi} \prod_{i \in \mathcal{I}} H^1(X, \Omega) \simeq \mathbb{G}_a^{\mathcal{I}} \xrightarrow{\mathrm{sum}} \mathbb{G}_a,$$

where the last arrow is the map  $(a_1, \dots, a_r) \rightarrow a_1 + \dots + a_r$ .

Write  $\mathcal{J}_{\psi}$  for the Artin-Schreier sheaf on  $\mathbb{G}_a$  corresponding to a fixed additive character  $\psi : \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_{\ell}^*$  and let  $\Psi_{\varpi}$  be the local system on  $\mathrm{Bun}_N^{\mathcal{F}_T}$  defined by

$$\Psi_{\varpi} := \mathrm{ev}_{\varpi}^*(\mathcal{J}_{\psi})[d_N],$$

where  $d_N = \dim \mathrm{Bun}_N^{\mathcal{F}_T}$ .

Let  $\overline{\Psi}_{\varpi}$  be the perverse sheaf on  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ , which is the Goresky-MacPherson extension of  $\Psi_{\varpi}$ :

$$\overline{\Psi}_{\varpi} := j_{!*}(\Psi_{\varpi}).$$

We also define *complexes* of perverse sheaves

$$\Psi_{\varpi!} := j_!(\Psi_{\varpi}), \quad \Psi_{\varpi*} := j_*(\Psi_{\varpi})$$

on  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ .

**Theorem 2.** *The canonical maps*

$$\Psi_{\varpi!} \rightarrow \overline{\Psi}_{\varpi} \rightarrow \Psi_{\varpi*}$$

*are isomorphisms.*

In other words, the  $*$ -restriction of  $\overline{\Psi}_{\varpi}$  to the complement of  $\mathrm{Bun}_N^{\mathcal{F}_T}$  in  $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$  vanishes.

4.2. **The definition of the category of sheaves.**

4.2.1. Next we reformulate Theorem 2 using the ind-stack  $\overline{\mathfrak{Bun}}_N^{\mathcal{F}_T}$ .

Let  $(\overline{x}, \overline{\nu})$  be as in Sect. 2.3 and consider the stack  $\overline{\mathfrak{Bun}}_N^{\mathcal{F}_T}$ . Denote by  $\text{pr}$  the natural projection  $\Lambda \rightarrow \Lambda_{G_{\text{ad}}}$ . Let us assume that for every  $k$ , the sum

$$\text{cond}_{x_k}(\varpi) + \text{pr}(\nu_k)$$

is still a dominant coweight of  $G_{\text{ad}}$ .

We have an identification of stacks

$$\overline{\mathfrak{Bun}}_N^{\mathcal{F}_T} \simeq \text{Bun}_N^{\mathcal{F}'_T},$$

where  $\mathcal{F}'_T := \mathcal{F}_T(-\sum_k \nu_k \cdot x_k)$ . The above condition on  $\varpi$  ensures that the corresponding meromorphic maps

$$\varpi'_i : \mathcal{L}_{\mathcal{F}'_T}^{\alpha_i} \rightarrow \Omega$$

are regular. Hence we obtain a morphism  $\text{ev}_{\varpi'} : \text{Bun}_N^{\mathcal{F}'_T} \rightarrow \mathbb{G}_a$ , and hence a morphism

$$\text{ev}_{\overline{\varpi}}^{\overline{x}, \overline{\nu}} : \overline{\mathfrak{Bun}}_N^{\mathcal{F}_T} \rightarrow \mathbb{G}_a.$$

Let  $\Psi_{\overline{\varpi}}^{\overline{x}, \overline{\nu}} = (\text{ev}_{\overline{\varpi}}^{\overline{x}, \overline{\nu}})^*(\mathcal{I}_{\psi})[d_N]$  be the corresponding local system on  $\overline{\mathfrak{Bun}}_N^{\mathcal{F}_T} \subset \overline{\mathfrak{Bun}}_N^{\mathcal{F}_T}$ . Let us denote by  $\overline{\Psi}_{\overline{\varpi}}^{\overline{x}, \overline{\nu}}$  be the Goresky-MacPherson extension of  $\Psi_{\overline{\varpi}}^{\overline{x}, \overline{\nu}}$ , and  $\Psi_{\overline{\varpi}}^{\overline{x}, \overline{\nu}}!$  and  $\Psi_{\overline{\varpi}}^{\overline{x}, \overline{\nu}}*$  be the corresponding complexes of perverse sheaves on  $\overline{\mathfrak{Bun}}_N^{\mathcal{F}_T}$ .

**4.2.2. Definition.** Let us fix data of  $\varpi$ , such that  $\text{cond}(\varpi) = 0$  (note that this means that we fix isomorphisms  $\mathcal{L}_{\mathcal{F}_T}^{\alpha_i} \simeq \Omega$ ). We define the *Whittaker category*  $\mathcal{W}_{\overline{\varpi}}^{\overline{x}}$  as the full abelian subcategory of the category of all perverse sheaves on  $\overline{\mathfrak{Bun}}_N^{\mathcal{F}_T}$ , which consists of the perverse sheaves whose irreducible subquotients are the sheaves  $\overline{\Psi}_{\overline{\varpi}}^{\overline{x}, \overline{\nu}}$ , with  $\overline{\nu}$  such that  $\nu_k \in \Lambda^{++}$  for all  $k = 1, \dots, n$ .

The following is the main result of this paper.

**Theorem 3.** (1) *The category  $\mathcal{W}_{\overline{\varpi}}^{\overline{x}}$  is semi-simple.*

(2) *The canonical maps  $\Psi_{\overline{\varpi}}^{\overline{x}, \overline{\nu}}! \rightarrow \overline{\Psi}_{\overline{\varpi}}^{\overline{x}, \overline{\nu}} \rightarrow \Psi_{\overline{\varpi}}^{\overline{x}, \overline{\nu}}*$  are isomorphisms.*

Theorem 3 will be proved in Sect. 6. As was already mentioned in Sect. 1, Theorem 2 will be obtained as a rather formal consequence of the explicit computation of the action of the Hecke functors on  $\mathcal{W}_{\overline{\varpi}}^{\overline{x}}$ , which will be introduced in the next section.

4.2.3. We claim that Theorem 3(2) and Theorem 2 are equivalent. Clearly, Theorem 3(2) is a particular case of Theorem 2. Let us explain how to derive Theorem 2 from Theorem 3(2).

Suppose first that the center  $Z(G)$  of  $G$  is connected. This means that the map  $\text{pr} : \Lambda \rightarrow \Lambda_{G_{\text{ad}}}$  is surjective. In this case, starting with an arbitrary  $\varpi$ , there exists a  $\Lambda$ -valued divisor  $D = \sum_k \mu_k \cdot x_k$  on  $X$ , such that  $\text{pr}(D) = \text{cond}(\varpi)$ .

Thus, if we set  $\mathcal{F}'_T := \mathcal{F}_T(D)$ , we will have  $\text{cond}(\varpi') = 0$  for the corresponding maps  $\varpi'_i : \mathcal{L}_{\mathcal{F}'_T}^{\alpha_i} \rightarrow \Omega$ . Moreover, we can then identify  $\overline{\mathfrak{Bun}}_N^{\mathcal{F}_T}$  with  $\overline{\mathfrak{Bun}}_N^{\mathcal{F}_T + \overline{\mu}}$ , and identify their closures in  $\overline{\mathfrak{Bun}}_N^{\mathcal{F}_T}$  and  $\overline{\mathfrak{Bun}}_N^{\mathcal{F}'_T}$ , respectively. The morphisms

$\text{ev}_\varpi : \overline{x}, \overline{\nu} \text{ Bun}_N^{\mathcal{F}_T} \rightarrow \mathbb{G}_a$  and  $\text{ev}_{\varpi'} : \overline{x}, \overline{\nu} + \overline{\mu} \text{ Bun}_N^{\mathcal{F}_T} \rightarrow \mathbb{G}_a$  coincide under this identification, as do the corresponding perverse sheaves.

Therefore we see that in the case when  $Z(G)$  is connected Theorem 2 and Theorem 3(2) are equivalent.

4.2.4. Let now  $G$  be an arbitrary reductive group, such that  $[G, G]$  is simply-connected, and consider the group  $G_1 := G \times_{Z(G)} T$ . Evidently,  $[G_1, G_1]$  is also simply-connected.

In addition,  $G_1$  has a connected center:  $Z(G_1) = T$  and Theorem 2 for  $G_1$  follows from Theorem 3.

Let  $T_1$  denote the Cartan subgroup of  $G_1$ ; we have an embedding  $T \hookrightarrow T_1$  and let  $\mathcal{F}_{T_1}$  denote the induced  $T_1$ -bundle over  $X$ .

We have a natural map of stacks  $\overline{\text{Bun}}_N^{\mathcal{F}_T} \rightarrow \overline{\text{Bun}}_{N_1}^{\mathcal{F}_{T_1}}$ , which is easily seen to be an isomorphism.

Moreover, the data of  $\varpi$  for  $G$  defines the corresponding data  $\varpi_1$  for  $G_1$  such that the sheaves  $\Psi_{\varpi!}$ ,  $\overline{\Psi}_\varpi$  and  $\Psi_{\varpi*}$  match with the corresponding perverse sheaves for  $G_1$ . Hence, Theorem 2 for  $G$  will follow from Theorem 3(2) for  $G_1$ .

## 5. HECKE FUNCTORS

### 5.1. The affine Grassmannian.

5.1.1. Let  $x \in X$  be an  $\overline{\mathbb{F}}_q$ -point,  $\mathcal{D}_x$  the formal disc around  $x$  in  $X$ , and  $t$  a formal coordinate at  $x$ . The choice of the formal coordinate allows us to identify  $\hat{\mathcal{O}}_x$  with  $\overline{\mathbb{F}}_q[[t]]$  and  $\hat{\mathcal{K}}_x$  with  $\overline{\mathbb{F}}_q((t))$ .

Recall the group scheme  $G(\hat{\mathcal{O}}_x)$  and the ind-group scheme  $G(\hat{\mathcal{K}}_x)$  that classify maps  $\mathcal{D}_x \rightarrow G$  and  $\mathcal{D}_x^\times \rightarrow G$ , respectively.

The affine Grassmannian  $\text{Gr}$  is an ind-scheme defined as the quotient  $G(\hat{\mathcal{K}}_x)/G(\hat{\mathcal{O}}_x)$ . In other words, for a scheme  $S$ ,  $\text{Hom}(S, \text{Gr})$  is the set of pairs  $(\mathcal{F}_G, \beta)$ , where  $\mathcal{F}_G$  is an  $S$ -family of  $G$ -bundles on  $\mathcal{D}_x$  and  $\beta$  is a trivialization of the corresponding family of bundles on  $\mathcal{D}_x^\times$ .

Alternatively, one can define  $\text{Gr}$  using the global curve  $X$ :  $\text{Hom}(S, \text{Gr})$  is a set of pairs  $(\mathcal{F}_G, \beta)$ , where  $\mathcal{F}_G$  is a  $G$ -bundle over  $X \times S$  and  $\beta$  is a trivialization of the restriction of  $\mathcal{F}_G$  to  $(X - x) \times S$ .

Evidently, to a data  $(\mathcal{F}_G, \beta)$  “over  $X$ ” one can attach a data  $(\mathcal{F}_G, \beta)$  “over  $\mathcal{D}_x$ ” and the fact that the two definitions coincide is a theorem due to Beauville and Laszlo [1].

5.1.2. The first description of  $\text{Gr}$  implies that it carries a natural action of  $G(\hat{\mathcal{K}}_x)$  and, in particular, of  $G(\hat{\mathcal{O}}_x)$ . For  $\lambda \in \Lambda^+$  let  $\text{Gr}^\lambda$  be the  $G(\hat{\mathcal{O}}_x)$ -orbit of the point  $\lambda(t) \cdot G(\hat{\mathcal{O}}_x) \in \text{Gr}$ , where  $\lambda(t) \in T(\hat{\mathcal{K}}_x) \subset G(\hat{\mathcal{K}}_x)$ . It is easy to see that  $\text{Gr}^\lambda$  is independent of the choice of  $T \subset G$  and of the uniformizer  $t \in \hat{\mathcal{O}}_x$ . Furthermore,  $\dim(\text{Gr}^\lambda) = \langle \lambda, 2\check{\rho} \rangle$  and  $\text{Gr}^\mu \subset \overline{\text{Gr}}^\lambda$  if and only if  $\lambda \geq \mu$ , i.e., when  $\lambda - \mu$  is a sum of positive roots of  ${}^L G$  – see, for example, [17]. The schemes  $\text{Gr}^\lambda$  form a stratification of the underlying reduced scheme of  $\text{Gr}$ .

### 5.2. The category $\text{P}_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$ .

5.2.1. Since the closures of  $G(\hat{\mathcal{O}}_x)$ -orbits on  $\text{Gr}$  are finite-dimensional, the abelian category  $\text{P}_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$  of  $G(\hat{\mathcal{O}}_x)$ -equivariant perverse sheaves on  $\text{Gr}$  is well-defined. By definition, every object of this category is supported over a finite union of schemes of the form  $\overline{\text{Gr}}^\lambda$ .

It is well-known that for every  $\overline{\mathbb{F}}_q$ -point of  $\text{Gr}$ , its stabilizer in  $G(\hat{\mathcal{O}}_x)$  is connected. Therefore, the irreducible objects in  $\text{P}_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$  are the Goresky–MacPherson extensions of constant local systems (appropriately shifted) on the strata  $\text{Gr}^\lambda$ ; we will denote these sheaves by  $\mathcal{A}_\lambda$ .

5.2.2. **Theorem.** (see [17]) *The category  $\text{P}_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$  is semi-simple.*

We remark that the argument given in Sect. 6.1 yields an alternative proof of this theorem.

5.2.3. Next we will describe the convolution operation  $\text{P}_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$ . This construction is nothing but a sheaf-theoretic analogue of the definition of the algebra structure on  $\mathbf{H}$ .

Consider the ind-scheme  $\text{Conv} := G(\hat{\mathcal{K}}_x) \times_{G(\hat{\mathcal{O}}_x)} G(\hat{\mathcal{K}}_x)/G(\hat{\mathcal{O}}_x)$ . It is easy to see that this scheme represents the following functor:  $\text{Hom}(S, \text{Conv})$  is the set of quadruples  $(\mathcal{F}_G^1, \beta^1, \mathcal{F}_G, \beta)$ , where  $\mathcal{F}_G^1$  and  $\mathcal{F}_G$  are  $G$ -torsors over  $\mathcal{D}_x$ ,  $\beta^1$  is an identification  $\mathcal{F}_G^1|_{\mathcal{D}_x^\times} \rightarrow \mathcal{F}_G|_{\mathcal{D}_x^\times}$  and  $\beta$  is a trivialization  $\mathcal{F}_G|_{\mathcal{D}_x^\times} \rightarrow \mathcal{F}_G^0|_{\mathcal{D}_x^\times}$ .

There is an obvious projection  $p_1 : \text{Conv} \rightarrow \text{Gr}$  which sends the quadruple  $(\mathcal{F}_G^1, \beta^1, \mathcal{F}_G, \beta)$  to  $(\mathcal{F}_G, \beta)$ ; it is the projection on the first factor:

$$G(\hat{\mathcal{K}}_x) \times_{G(\hat{\mathcal{O}}_x)} G(\hat{\mathcal{K}}_x)/G(\hat{\mathcal{O}}_x) \rightarrow G(\hat{\mathcal{K}}_x)/G(\hat{\mathcal{O}}_x) = \text{Gr}.$$

Moreover, this projection realizes  $\text{Conv}$  as a fibration over  $\text{Gr}$  with a typical fiber isomorphic again to  $\text{Gr}$ .

More precisely,  $\text{Conv}$  is a bundle associated to the  $G(\hat{\mathcal{O}}_x)$ -scheme  $\text{Gr}$  and  $G(\hat{\mathcal{O}}_x)$ -torsor  $G(\hat{\mathcal{K}}_x) \rightarrow G(\hat{\mathcal{K}}_x)/G(\hat{\mathcal{O}}_x) = \text{Gr}$ .

Thus, starting from a perverse sheaf  $\mathcal{S}_1$  on  $\text{Gr}$  and a  $G(\hat{\mathcal{O}}_x)$ -equivariant perverse sheaf  $\mathcal{S}_2$  on  $\text{Gr}$  we can construct their twisted external product  $\mathcal{S}_1 \tilde{\boxtimes} \mathcal{S}_2$ , which is a perverse sheaf on  $\text{Conv}$  (see Sect. 1.4).

Note that the product map gives rise to a second projection

$$\text{Conv} = G(\hat{\mathcal{K}}_x) \times_{G(\hat{\mathcal{O}}_x)} G(\hat{\mathcal{K}}_x)/G(\hat{\mathcal{O}}_x) \rightarrow G(\hat{\mathcal{K}}_x)/G(\hat{\mathcal{O}}_x) = \text{Gr}.$$

On the level of the corresponding functors, this map sends a quadruple  $(\mathcal{F}_G^1, \beta^1, \mathcal{F}_G, \beta)$  as above to  $(\mathcal{F}_G^1, \beta')$ , where  $\beta'$  is obtained as a composition  $\beta \circ \beta^1$ . We will denote this projection by  $p_2$ .

For two perverse sheaves  $\mathcal{S}_1$  and  $\mathcal{S}_2$  on  $\mathcal{G}$  with  $\mathcal{S}_2$  being  $G(\hat{\mathcal{O}}_x)$ -equivariant, we denote by  $\mathcal{S}_1 \star \mathcal{S}_2$  the complex  $p_{2!}(\mathcal{S}_1 \tilde{\boxtimes} \mathcal{S}_2)$ . By construction, if  $\mathcal{S}_1$  is also  $G(\hat{\mathcal{O}}_x)$ -equivariant, then so is  $\mathcal{S}_1 \star \mathcal{S}_2$ .

**5.2.4. Theorem.** ([17]) *For  $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{P}_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$ , the complex  $\mathcal{S}_1 \star \mathcal{S}_2$  is a perverse sheaf.*

Actually, even more is true:  $\mathcal{S}_1 \star \mathcal{S}_2$  is a perverse sheaf for  $\mathcal{S}_1 \in \mathcal{P}_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$  and any perverse sheaf  $\mathcal{S}_2$ . However, we will not need this stronger statement.

**5.2.5.** Thus,  $\mathcal{S}_1, \mathcal{S}_2 \mapsto \mathcal{S}_1 \star \mathcal{S}_2$  defines a binary operation on  $\mathcal{P}_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$ . It is easy to see from the definition of  $\star$  that it admits a natural associativity constraint, which makes  $\mathcal{P}_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$  into a monoidal category.

In addition, the inversion  $g \mapsto g^{-1}$  on  $G(\hat{\mathcal{K}}_x)$  induces a covariant self-functor on  $\mathcal{P}_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$ , which we will denote by  $\mathcal{S} \mapsto \ast \mathcal{S}$ .

The following theorem can be viewed as a sheaf-theoretic version of the Satake isomorphism (see Theorem 1.1.2).

**5.2.6. Theorem.** *The category  $\mathcal{P}_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$  admits a commutativity constraint, i.e., it is a tensor category. Moreover, as such, it is equivalent to the category  $\mathrm{Rep}({}^L G)$  of finite-dimensional representations of  ${}^L G$  in such a way that*

- (1) *The object  $\mathcal{A}_\lambda \in \mathcal{P}_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$  goes to  $V^\lambda$ .*
- (2) *Let  $\mathbb{D}$  be the Verdier duality functor on  $\mathcal{P}_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$ . The functor  $\mathcal{S} \rightarrow \mathbb{D}(\ast \mathcal{S})$  corresponds to the contragredient duality functor.*

**5.2.7. Remark.** To be precise, this result has been proved in [10, 18] over the ground field  $\mathbb{C}$  (in this setting, this isomorphism was conjectured by V. Drinfeld; see also [17]). But the proof outlined in [18] can be generalized to the case of the ground field  $\overline{\mathbb{F}}_q$ ; see also [20].

### 5.3. The Hecke action on ${}_{x,\infty}\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ .

**5.3.1. Definition.** For a point  $x \in X$  the *Hecke correspondence* stack  $\mathcal{H}_x$  is defined as follows. For an  $\overline{\mathbb{F}}_q$ -scheme  $S$ ,  $\mathrm{Hom}(S, \mathcal{H}_x)$  is a groupoid, whose objects are triples  $(\mathcal{F}_G, \mathcal{F}'_G, \beta)$ , where  $\mathcal{F}_G$  and  $\mathcal{F}'_G$  are  $G$ -bundles over  $X \times S$  and  $\beta$  is an isomorphism

$$\beta : \mathcal{F}_G|_{(X-x) \times S} \simeq \mathcal{F}'_G|_{(X-x) \times S}.$$

The definition of morphisms in  $\mathrm{Hom}(S, \mathcal{H}_x)$  is straightforward.

Let  $h^\leftarrow$  and  $h^\rightarrow$  be the projections  $\mathcal{H}_x \rightarrow \mathrm{Bun}_G$ , sending  $(\mathcal{F}_G, \mathcal{F}'_G, \beta)$  to  $\mathcal{F}_G$  and  $\mathcal{F}'_G$ , respectively.

**5.3.2.** Recall the  $G(\hat{\mathcal{O}}_x)$ -bundle  $\mathcal{G}_x$  over  $\mathrm{Bun}_G$ . It follows from the definitions that each of the projections  $h^\leftarrow$  and  $h^\rightarrow$  realizes  $\mathcal{H}_x$  as a fibration over  $\mathrm{Bun}_G$  attached to  $\mathcal{G}_x$  and the  $G(\hat{\mathcal{O}}_x)$ -scheme  $\mathrm{Gr}$ :

$$(5.1) \quad \mathcal{H}_x \simeq \mathcal{G}_x \times_{G(\hat{\mathcal{O}}_x)} \mathrm{Gr}.$$

Thus, if  $\mathcal{S}$  is an object of  $P_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$ , we can use either  $h^\rightarrow$  or  $h^\leftarrow$  to produce objects  $\mathcal{S}^l$  and  $\mathcal{S}^r$  in  $\text{Sh}(\mathcal{H}_x)$  by taking the twisted external product of  $\mathcal{S}$  with the constant sheaf on  $\text{Bun}_G$ . Note that there is a natural isomorphism

$$(5.2) \quad (*\mathcal{S})^l \simeq \mathcal{S}^r.$$

Let  $\text{Sh}(\text{Bun}_G)$  be the derived category of  $\overline{\mathbb{Q}}_\ell$ -sheaves on  $\text{Bun}_G$ . It is well-known that  $P_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$  “acts” on  $\text{Sh}(\text{Bun}_G)$  by convolution (this action is used in the definition of “Hecke eigensheaves” in the geometric Langlands correspondence). We will need an analogous action of  $P_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$  on the derived category  $\text{Sh}_{(x,\infty)}(\overline{\text{Bun}}_N^{\mathcal{F}_T})$  of  $\overline{\mathbb{Q}}_\ell$ -sheaves on  $_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$ .

5.3.3. Let  $Z$  denote the fiber product

$$(5.3) \quad Z = \mathcal{H}_x \times_{\text{Bun}_G} {}_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T},$$

where the morphism  $\mathcal{H}_x \rightarrow \text{Bun}_G$  that we use in the above formula is the projection  $h^\rightarrow$ .

In other words,  $Z$  is a fibration over  $_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  that corresponds to the  $G(\hat{\mathcal{O}}_x)$ -torsor  $\mathcal{G}_x|_{_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}}$  and the  $G(\hat{\mathcal{O}}_x)$ -scheme  $\text{Gr}$ .

5.3.4. **Proposition.** *There exists a second projection  $Z \xrightarrow{{}'h^\leftarrow} {}_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  which renders the diagram*

$$(5.4) \quad \begin{array}{ccccc} {}_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T} & \xleftarrow{{}'h^\leftarrow} & Z & \xrightarrow{{}'h^\rightarrow} & {}_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T} \\ \bar{p} \downarrow & & \bar{p} \downarrow & & \bar{p} \downarrow \\ \text{Bun}_G & \xleftarrow{h^\leftarrow} & \mathcal{H}_x & \xrightarrow{h^\rightarrow} & \text{Bun}_G \end{array}$$

*commutative. Moreover, the left square in the above diagram is Cartesian as well.*

5.3.5. *Proof.* By definition, the stack  $_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$  classifies pairs  $(\mathcal{F}_G, \kappa)$ , where  $\kappa$  is a collection of maps

$$\kappa^{\tilde{\lambda}} : \mathcal{L}_{\mathcal{F}_T}^{\tilde{\lambda}} \rightarrow \mathcal{V}_{\mathcal{F}'_G}^{\tilde{\lambda}}(\infty \cdot x)$$

subject to the Plücker relations.

Therefore, the stack  $Z$  classifies triples  $(\mathcal{F}_G, \mathcal{F}'_G, \kappa, \beta)$ , with  $\mathcal{F}_G$  and  $\kappa$  as above and  $\beta$  is an isomorphism  $\mathcal{F}_G|_{X-x} \simeq \mathcal{F}'_G|_{X-x}$ .

Hence, from  $\kappa$  and  $\beta$  we obtain a system of maps

$$\kappa'^{\tilde{\lambda}} : \mathcal{L}_{\mathcal{F}_T}^{\tilde{\lambda}} \rightarrow \mathcal{V}_{\mathcal{F}'_G}^{\tilde{\lambda}}(\infty \cdot x),$$

which satisfy the Plücker relations.

Let us set  $'h^\leftarrow(\mathcal{F}_G, \mathcal{F}'_G, \kappa, \beta) = (\mathcal{F}_G, \kappa')$ , with the collection  $\kappa' = \{\kappa'^{\tilde{\lambda}}\}$  defined above. It is clear that the left square of the above diagram is commutative and Cartesian.

5.3.6. The morphism  $'h^\rightarrow$  realizes  $Z$  as a fibration  $\mathcal{G}_x \times_{G(\hat{\mathcal{O}}_x)} \text{Gr}$ . Therefore given  $\mathcal{S} \in \text{P}_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$  and  $\mathcal{T} \in \text{Sh}_{(x,\infty)\overline{\text{Bun}}_N^{\mathcal{F}_T}}$ , using the notation introduced in Sect. 1.4, we can form the twisted tensor product  $(\mathcal{T} \boxtimes \mathcal{S})^r$ . Note that because  $'h^\leftarrow$  is proper on the support of  $(\mathcal{T} \boxtimes \mathcal{S})^r$ , we have

$$'h_*^\leftarrow((\mathcal{T} \boxtimes \mathcal{S})^r) = 'h_!^\leftarrow((\mathcal{T} \boxtimes \mathcal{S})^r).$$

We then set:

$$(5.5) \quad \mathcal{T} \star \mathcal{S} := 'h_!^\leftarrow((\mathcal{T} \boxtimes \mathcal{S})^r) = 'h_*^\leftarrow((\mathcal{T} \boxtimes \mathcal{S})^r).$$

Analogously, considering  $Z$  as a fibration  $\mathcal{G}_x \times_{G(\hat{\mathcal{O}}_x)} \text{Gr}$  corresponding to the morphism  $'h^\leftarrow$ , we can construct the twisted tensor product  $(\mathcal{T} \boxtimes \mathcal{S})^l$ . Then we obtain a functor “from left to right”:

$$\mathcal{T}, \mathcal{S} \mapsto \mathcal{S} \star \mathcal{T} = 'h_!^\rightarrow((\mathcal{T} \boxtimes (*\mathcal{S}))^l).$$

5.3.7. *Properties of the action.* The Hecke functors defined above

$$\text{Perv}_{G(\hat{\mathcal{O}}_x)}(\text{Gr}) \times \text{Sh}_{(x,\infty)\overline{\text{Bun}}_N^{\mathcal{F}_T}} \rightarrow \text{Sh}_{(x,\infty)\overline{\text{Bun}}_N^{\mathcal{F}_T}}$$

are compatible with the tensor structure on  $\text{Perv}_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$  in the following sense:

5.3.8. **Lemma.** *For any  $\mathcal{S}_1, \mathcal{S}_2 \in \text{Perv}_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$  we have functorial isomorphisms*

$$\begin{aligned} (\mathcal{T} \star \mathcal{S}_1) \star \mathcal{S}_2 &\simeq \mathcal{T} \star (\mathcal{S}_1 \star \mathcal{S}_2), \\ \mathcal{S}_1 \star (\mathcal{S}_2 \star \mathcal{T}) &\simeq (\mathcal{S}_1 \star \mathcal{S}_2) \star \mathcal{T}, \end{aligned}$$

*such that the pentagon identity holds for three-fold convolution products.*

Moreover, we also have the following proposition which follows in a straightforward way from the definitions..

5.3.9. **Proposition.**

- (1) *The Hecke functors commute with Verdier duality in the sense that  $\mathbb{D}(\mathcal{S} \star \mathcal{T}) \simeq \mathbb{D}(\mathcal{S}) \star \mathbb{D}(\mathcal{T})$ , and  $\mathbb{D}(\mathcal{T} \star \mathcal{S}) \simeq \mathbb{D}(\mathcal{T}) \star \mathbb{D}(\mathcal{S})$ .*
- (2) *For a fixed  $\mathcal{S} \in \text{P}_{G(\hat{\mathcal{O}}_x)}(\text{Gr})$ , the functors  $\mathcal{T} \mapsto \mathcal{T} \star \mathcal{S}$  and  $\mathcal{T} \mapsto \mathbb{D}(\mathcal{S}) \star \mathcal{T}$  from  $\text{Sh}_{(x,\infty)\overline{\text{Bun}}_N^{\mathcal{F}_T}}$  to itself are mutually both left and right adjoint.*
- (3) *There is a functorial isomorphism  $\mathcal{T} \star \mathcal{S} \simeq (*\mathcal{S}) \star \mathcal{T}$ .*

5.4. **Action on the canonical sheaves.**

5.4.1. Now we take as  $\mathcal{S}$ , the perverse sheaf  $\mathcal{A}_\lambda$ , which is the Goresky–MacPherson extension of the constant sheaf (appropriately shifted) on  $\mathrm{Gr}^\lambda$ . Note that we have:  $*\mathcal{A}_\lambda = \mathcal{A}_{-w_0(\lambda)}$ .

The following statement, which will be proved in Sect. 7, is a crucial step in our proof of Theorem 3(1).

**Theorem 4.** *Suppose that  $\mathrm{cond}_x(\varpi) = 0$ . Then*

$$\overline{\Psi}_\varpi^{x,0} \star \mathcal{A}_\lambda \simeq \overline{\Psi}_\varpi^{x,\lambda}.$$

5.4.2. According to Lemma 5.3.8 and Theorem 5.2.6, for any  $\mu \in \Lambda^+$ , there exists a functorial isomorphism

$$(\mathcal{T} \star \mathcal{A}_\lambda) \star \mathcal{A}_\mu \simeq \oplus_{\nu \in \Lambda^+} (\mathcal{T} \star \mathcal{A}_\nu) \otimes \mathrm{Hom}_{L_G}(V^\nu, V^\lambda \otimes V^\mu).$$

Therefore we obtain:

5.4.3. **Corollary.** *Suppose that  $\mathrm{cond}_x(\varpi) = 0$ . Then*

$$(5.6) \quad \overline{\Psi}_\varpi^{x,\mu} \star \mathcal{A}_\lambda \simeq \oplus_{\nu \in \Lambda^+} \overline{\Psi}_\varpi^{x,\nu} \otimes \mathrm{Hom}_{L_G}(V^\nu, V^\lambda \otimes V^\mu)$$

5.4.4. *The category  $P_{N(\hat{\mathcal{K}}_x)}^\chi(\mathrm{Gr})$ .* Let us explain why Theorem 3 and Theorem 4 allow us to construct a category  $P_{N(\hat{\mathcal{K}})}^\chi(\mathrm{Gr})$  which satisfies the properties listed in Sect. 1.2.4.

Choose data of  $\varpi$  with  $\mathrm{cond}(\varpi) = 0$  (in fact,  $\mathrm{cond}_x(\varpi) = 0$  would suffice). Set

$$P_{N(\hat{\mathcal{K}})}^\chi(\mathrm{Gr}) := \mathcal{W}_\varpi^x.$$

Then Theorem 4 implies that the Hecke action of  $P_{G(\hat{\mathcal{O}}_x)}(\mathrm{Gr})$  on  $\mathrm{Sh}(x, \infty \overline{\mathrm{Bun}}_N^{\mathcal{F}_T})$  preserves  $\mathcal{W}_\varpi^x$  and that  $\overline{\Psi}_\varpi^{x,0} \star \mathcal{A}_\lambda \simeq \overline{\Psi}_\varpi^{x,\lambda}$ .

This gives us the first two properties of  $P_{N(\hat{\mathcal{K}})}^\chi(\mathrm{Gr})$ . The third property is insured by Theorem 3.

We will see in Sect. 8 that the computation of stalks of  $\overline{\Psi}_\varpi^{x,\mu+\nu} \star \mathcal{A}_\lambda$  is equivalent to the computation of the cohomology (1.7).

Our plan now is the following. In Sect. 6 we derive Theorem 3 from Theorem 4. Then in Sect. 7 we prove Theorem 4. Finally, we use these results in Sect. 8 to prove Theorem 1.

## 6. PROOF OF THEOREM 3

### 6.1. Proof of Theorem 3(1).

6.1.1. To simplify notation we will assume that  $\overline{x}$  consists of one point  $x$ . The proof in the general case is exactly the same.

To show that  $\mathcal{W}_\varpi^x$  is semi-simple, it suffices to prove that

$$\mathrm{Ext}^1(\overline{\Psi}_\varpi^{x,\mu}, \overline{\Psi}_\varpi^{x,\nu}) = 0$$

for any  $\mu, \nu \in \Lambda^{++}$ .

6.1.2. **Step 1.** We claim that it is enough to prove that

$$(6.1) \quad \text{Ext}^1(\overline{\Psi}_{\varpi}^{x,0}, \overline{\Psi}_{\varpi}^{x,\nu}) = 0.$$

Indeed, using Theorem 4 and Proposition 5.3.9(2,3) we obtain:

$$\text{Ext}^1(\overline{\Psi}_{\varpi}^{x,\mu}, \overline{\Psi}_{\varpi}^{x,\lambda}) \simeq \text{Ext}^1(\overline{\Psi}_{\varpi}^{x,0} \star \mathcal{A}_{\mu}, \overline{\Psi}_{\varpi}^{x,\lambda}) \simeq \text{Ext}^1(\overline{\Psi}_{\varpi}^{x,0}, \overline{\Psi}_{\varpi}^{x,\lambda} \star \mathcal{A}_{-w_0(\mu)}),$$

which is a direct sum of terms of the form  $\text{Ext}^1(\overline{\Psi}_{\varpi}^{x,0}, \overline{\Psi}_{\varpi}^{x,\nu}), \nu \in \Lambda^+$ , according to Corollary 5.4.3.

6.1.3. **Step 2.** We claim that it suffices to show that for all  $\lambda \in \Lambda^+$ ,

$$(6.2) \quad \text{Ext}^1(\overline{\Psi}_{\varpi}^{x,\lambda}, \overline{\Psi}_{\varpi}^{x,\lambda}) = 0.$$

First of all, (6.1) is obvious unless  $\nu \in \text{Span}\{\alpha_i\}$ , for otherwise  $\overline{\Psi}_{\varpi}^{x,0}$  and  $\overline{\Psi}_{\varpi}^{x,\nu}$  live on different connected components of  ${}_{x,\infty}\overline{\text{Bun}}_N^{\mathcal{F}_T}$ . Hence we can assume that  $\nu \in \text{Span}\{\alpha_i\}$ , which means that  $Z({}^L G)$  acts trivially on  $V^{\nu}$ .

The following result is well-known.

**Lemma.** *Let  $H$  be a reductive algebraic group over an algebraically closed field of characteristic 0 and  $V$  be an irreducible  $H$ -module such that  $Z(H)$  acts trivially on  $V$ . Then there exists another irreducible  $H$ -module  $W$ , such that  $V$  is a direct summand in  $W \otimes W^*$ .*

This lemma, combined with Corollary 5.4.3, shows that for  $\nu \in \Lambda^{++}$ , there exists  $\lambda \in \Lambda^{++}$ , such that  $\overline{\Psi}_{\varpi}^{x,\nu}$  is a direct summand in  $\overline{\Psi}_{\varpi}^{x,\lambda} \star \mathcal{A}_{-w_0(\lambda)}$ .

Using Step 1, it is therefore enough to show that

$$\text{Ext}^1(\overline{\Psi}_{\varpi}^{x,0}, \overline{\Psi}_{\varpi}^{x,\lambda} \star \mathcal{A}_{-w_0(\lambda)})$$

for every  $\lambda \in \Lambda^{++}$ . But Proposition 5.3.9(2,3) and Theorem 4 imply:

$$(6.3) \quad \text{Ext}^1(\overline{\Psi}_{\varpi}^{x,0}, \overline{\Psi}_{\varpi}^{x,\lambda} \star \mathcal{A}_{-w_0(\lambda)}) \simeq \text{Ext}^1(\overline{\Psi}_{\varpi}^{x,\lambda}, \overline{\Psi}_{\varpi}^{x,\lambda}).$$

6.1.4. **Step 3.** Finally, to prove (6.2) let us recall the following well-known property of the Goresky-MacPherson extension.

**Lemma.** *Let  $Y_0 \xrightarrow{j} Y$  be an embedding of algebraic stacks and let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be two perverse sheaves on  $Y_0$ . Then the restriction map*

$$\text{Ext}_Y^1(j_{!*}(\mathcal{S}_1), j_{!*}(\mathcal{S}_2)) \rightarrow \text{Ext}_{Y_0}^1(\mathcal{S}_1, \mathcal{S}_2)$$

*is injective.*

Hence, it is enough to prove that

$$(6.4) \quad \text{Ext}_{\lambda,x \text{ Bun}_N^{\mathcal{F}_T}}^1(\overline{\Psi}_{\varpi}^{x,\lambda}, \overline{\Psi}_{\varpi}^{x,\lambda}) = 0.$$

Recall from Sect. 4.2.3 that we can identify  ${}_{\lambda,x}\mathrm{Bun}_N^{\mathcal{F}_T}$  with  ${}_{0,x}\mathrm{Bun}_N^{\mathcal{F}'_T} = \mathrm{Bun}_N^{\mathcal{F}'_T}$  for  $\mathcal{F}'_T = \mathcal{F}_T(-\lambda \cdot x)$ . Moreover, there exist  $\varpi'_i : \mathcal{L}_{\mathcal{F}'_T}^{\alpha_i} \rightarrow \Omega$  of conductor  $\lambda \cdot x$ , such that under this identification the sheaf  $\Psi_{\varpi}^{x,\lambda}$  becomes  $\Psi_{\varpi'}^{x,0} = \Psi_{\varpi'}$ . Therefore (6.4) is equivalent to

$$(6.5) \quad \mathrm{Ext}_{\mathrm{Bun}_N^{\mathcal{F}'_T}}^1(\Psi_{\varpi'}, \Psi_{\varpi'}) = 0.$$

Since  $\Psi_{\varpi'}$  is a rank 1 local system on  $\mathrm{Bun}_N^{\mathcal{F}'_T}$ , (6.5) is equivalent to

$$\mathrm{Ext}_{\mathrm{Bun}_N^{\mathcal{F}'_T}}^1(\overline{\mathbb{Q}}_{\ell}, \overline{\mathbb{Q}}_{\ell}) = 0.$$

According to Corollary 3.2.4, for  $k$  large,  $\mathrm{Bun}_N^{\mathcal{F}'_T} \simeq {}^k\mathcal{N}^{\epsilon}/N'_k$  and the space  ${}^k\mathcal{N}_y^{\epsilon}$  is isomorphic to a tower of affine spaces. Hence

$$\mathrm{Ext}_{\mathrm{Bun}_N^{\mathcal{F}'_T}}^1(\overline{\mathbb{Q}}_{\ell}, \overline{\mathbb{Q}}_{\ell}) = \mathrm{Ext}_{{}^k\mathcal{N}^{\epsilon}}^1(\overline{\mathbb{Q}}_{\ell}, \overline{\mathbb{Q}}_{\ell}) = H^1({}^k\mathcal{N}^{\epsilon}) = 0,$$

where the first isomorphism follows from the fact that  $N'_k$  is connected.

This completes the proof of Theorem 3(1).

## 6.2. Proof of Theorem 3(2).

6.2.1. The key step in proving part (2) of Theorem 3 is the following proposition.

**Proposition 6.2.1.** (1) *The complex  $\Psi_{\varpi}^{\overline{x},\overline{\nu}}$  is a perverse sheaf.*

(2)  *$\Psi_{\varpi}^{\overline{x},\overline{\nu}}$  belongs to the category  $\mathcal{W}_{\varpi}^{\overline{x}}$ .*

Let  $j : Y_0 \hookrightarrow Y$  be an affine embedding, and  $\mathcal{E}$  a local system on  $Y_0$ . Then the sheaf  $j_!(\mathcal{E})$  is perverse sheaf.

Therefore statement (1) of Proposition 6.2.1 follows from Proposition 3.3.1.

Now we turn to part (2) of Proposition 6.2.1.

6.2.2. Note that  $\overline{{}_{\overline{x},\overline{\nu}}\mathrm{Bun}_N^{\mathcal{F}_T}}$  is the union of the strata  $\overline{{}_{\overline{x} \cup \overline{z}, \overline{\mu} \cup \overline{\lambda}}\mathrm{Bun}_N^{\mathcal{F}_T}}$ , where  $\mu_i \leq \nu_i, i = 1, \dots, n, \overline{z} = \{z_1, \dots, z_m\}, \overline{\lambda} = \{\lambda_1, \dots, \lambda_m\}$ , and  $\lambda_j \in -\Lambda^+, j = 1, \dots, m$ .

Call a stratum *relevant* if  $\overline{\lambda} = 0$  and  $\mu_i \in \Lambda^{++}$  for all  $i$  and irrelevant otherwise. Proposition 6.2.1(2) will be derived from the following statement.

### 6.2.3. Lemma.

(1) *The  $*$ -restriction of  $\overline{\Psi}_{\varpi}^{\overline{x},\overline{\nu}}$  to any irrelevant stratum is 0.*

(2) *The perverse cohomologies of the  $*$ -restriction of  $\overline{\Psi}_{\varpi}^{\overline{x},\overline{\nu}}$  to a relevant stratum  $\overline{{}_{\overline{x},\overline{\mu}}\mathrm{Bun}_N^{\mathcal{F}_T}}$  are direct sums of copies of  $\Psi_{\varpi}^{\overline{x},\overline{\mu}}$ .*

Replacing  $\mathcal{F}_T$  by

$$\mathcal{F}'_T = \mathcal{F}_T \left( - \sum_{i=1}^n \nu_i \cdot x_i \right)$$

transforms the sheaf  $\overline{\Psi}_{\varpi}^{\overline{x},\overline{\nu}}$  over  $\overline{{}_{\overline{x},\overline{\nu}}\mathrm{Bun}_N^{\mathcal{F}_T}}$  to  $\overline{\Psi}_{\varpi'}$  over  $\mathrm{Bun}_N^{\mathcal{F}'_T}$ , where  $\varpi'$  has conductor  $\sum_i \mathrm{pr}(\nu_i) \cdot x_i$ . Thus Lemma 6.2.3 can be reformulated as follows:

**6.2.4. Lemma.** *The perverse cohomologies of the  $*$ -restriction of  $\overline{\Psi}_{\varpi'}$  to a stratum of the form  $\overline{x, \mu} \text{Bun}_N^{\mathcal{F}_T}$  (where  $\mu_i \in -\Lambda^+$ ) are zero unless  $\text{pr}(\mu_i) + \text{cond}_{x_i}(\varpi') \in \Lambda_{G_{\text{ad}}}^{++}$  for every  $i$ . In the latter case, they are direct sums of copies of  $\Psi_{\varpi'}^{\overline{x, \mu}}$ .*

**6.2.5.** Let  $y$  be a point different from the  $x_i$ 's and recall that we have a sequence of inclusions:

$$\overline{x, \mu} \text{Bun}_N^{\mathcal{F}_T} \hookrightarrow {}_{y,0} \widetilde{\text{Bun}}_N^{\mathcal{F}_T} \hookrightarrow \overline{\text{Bun}}_N^{\mathcal{F}_T}.$$

Recall also the  $N(\hat{\mathcal{O}}_y)$ -bundles  $\mathcal{N}_y^\epsilon$  and  $\widetilde{\mathcal{N}}_y^\epsilon$  over  $\text{Bun}_N^{\mathcal{F}_T}$  and  ${}_{y,0} \widetilde{\text{Bun}}_N^{\mathcal{F}_T}$ , respectively. We will denote the restriction of  $\widetilde{\mathcal{N}}_y^\epsilon$  to the stratum  $\overline{x, \mu} \text{Bun}_N^{\mathcal{F}_T}$  by  $\overline{x, \mu} \mathcal{N}_y^\epsilon$ . In addition, we will denote by  ${}^k \mathcal{N}_y^\epsilon$ ,  ${}^k \widetilde{\mathcal{N}}_y^\epsilon$  and  ${}^k_{\overline{x, \mu}} \mathcal{N}_y^\epsilon$  the corresponding  $N'_k$ -bundles.

As shown in Sect. 3.2.6, the stacks  ${}^k \mathcal{N}_y^\epsilon$ ,  ${}^k \widetilde{\mathcal{N}}_y^\epsilon$  and  ${}^k_{\overline{x, \mu}} \mathcal{N}_y^\epsilon$  carry an action of a bigger group, namely,  $N_k$ . The proof of Lemma 6.2.4 will be obtained by analyzing this action.

**6.2.6.** *The additive character of  $N(\hat{\mathcal{K}}_y)$ .* The data of  $\epsilon$  and  $\varpi'$  give rise to a homomorphism  $\chi : N(\hat{\mathcal{K}}_y) \rightarrow \mathbb{G}_a$ . Indeed, consider the ind-group scheme  $\Omega_{\hat{\mathcal{K}}_y} = \Gamma(\mathcal{D}_y^\times, \Omega)$ . The residue map gives rise to a homomorphism  $\text{Res} : \Omega_{\hat{\mathcal{K}}_y} \rightarrow \mathbb{G}_a$ . The additive character  $\chi$  is defined as the sum

$$\chi = \sum_{i \in \mathcal{I}} {}^i \chi,$$

where  ${}^i \chi$  is defined as the composition

$$N(\hat{\mathcal{K}}_y) \rightarrow N/[N, N](\hat{\mathcal{K}}_y) \xrightarrow{i} \mathbb{G}_a(\hat{\mathcal{K}}_y) \xrightarrow{\varpi' \circ \epsilon} \Omega_{\hat{\mathcal{K}}_y} \xrightarrow{\text{Res}} \mathbb{G}_a.$$

It is easy to see that  $\chi$  vanishes on  $N(\hat{\mathcal{O}}_y)$ . Therefore, for each  $k > 0$ , we obtain an additive character  $\chi^k : N_k \rightarrow \mathbb{G}_a$ .

In what follows, given an  $N_k$ -stack  $Y$  and a morphism  $Y \rightarrow \mathbb{G}_a$ , we will say that the latter is  $(N_k, \chi^k)$ -equivariant if the diagram

$$\begin{array}{ccc} N_k \times Y & \longrightarrow & N_k \times \mathbb{G}_a \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathbb{G}_a \end{array}$$

is commutative, where the right vertical arrow is the composition of  $\chi$  and the addition in  $\mathbb{G}_a$ .

If  $Y$  is an  $N_k$ -stack and  $\mathcal{S}$  is an object of  $\text{Sh}(Y)$ , we shall say that  $\mathcal{S}$  is  $(N_k, \chi^k)$ -equivariant if its pull-back under the action map  $N_k \times Y \rightarrow Y$  is isomorphic to  $\chi^{k*}(\mathcal{J}_\psi) \boxtimes \mathcal{S}$  so that the natural associativity requirement holds (recall that  $\mathcal{J}_\psi$  denotes the Artin-Schreier sheaf on  $\mathbb{G}_a$ ).

Evidently, if  $Y \rightarrow \mathbb{G}_a$  is an  $(N_k, \chi^k)$ -equivariant morphism, then the pull-back of  $\mathcal{J}_\psi$  on  $Y$  with respect to this map is  $(N_k, \chi^k)$ -equivariant as a complex.

6.2.7. *The  $N(\hat{\mathcal{K}}_y)$ -equivariance.* The crucial observation is that the map

$${}^k\mathcal{N}_y^\epsilon \rightarrow \mathrm{Bun}_N^{\mathcal{F}_T'} \xrightarrow{\mathrm{ev}_{\varpi'}} \mathbb{G}_a$$

is  $(N_k, \chi^k)$ -equivariant. This follows from the definition of  $\chi^k$ .

Hence, the pull-back of  $\Psi_{\varpi'}$  under the natural projection  ${}^k\mathcal{N}_y^\epsilon \rightarrow \mathrm{Bun}_N^{\mathcal{F}_T'}$  is an  $(N_k, \chi^k)$ -equivariant sheaf on  ${}^k\mathcal{N}_y^\epsilon$ . Therefore, by functoriality, the pull-back of  $\bar{\Psi}_{\varpi'}$  to  ${}^k\tilde{\mathcal{N}}_y^\epsilon$  and the perverse cohomologies of the  $*$ -restriction of the latter to  $\frac{k}{\bar{x}, \bar{\mu}}\mathcal{N}_y^\epsilon$  are  $(N_k, \chi^k)$ -equivariant.

Therefore, Lemma 6.2.4 is a consequence of the following general result on equivariant sheaves on  $\frac{k}{\bar{x}, \bar{\mu}}\mathcal{N}_y^\epsilon$ :

**6.2.8. Lemma.** *Let  $\mathcal{S}$  be an  $(N_k, \chi^k)$ -equivariant perverse sheaf on  $\frac{k}{\bar{x}, \bar{\mu}}\mathcal{N}_y^\epsilon$ . Assume in addition that  $k$  is large enough. Then  $\mathcal{S}$  vanishes unless for every  $i$ ,  $\mathrm{pr}(\mu_i) + \mathrm{cond}_{x_i}(\varpi') \in \Lambda_{G_{\mathrm{ad}}}^{++}$ . In the latter case,  $\mathcal{S}$  is a direct sum of copies of the pull-back of  $\Psi_{\varpi'}^{\bar{x}, \bar{\mu}}$  from  $\frac{k}{\bar{x}, \bar{\mu}}\mathrm{Bun}_N^{\mathcal{F}_T'}$  to  $\frac{k}{\bar{x}, \bar{\mu}}\mathcal{N}_y^\epsilon$  (appropriately shifted).*

6.2.9. *Proof of Lemma 6.2.8.* We know from Proposition 3.1.4 that for  $k$  large enough  $\frac{k}{\bar{x}, \bar{\mu}}\mathcal{N}_y^\epsilon$  is a scheme and can be identified with the quotient  $N_{\mathrm{out}, k}^{\mathcal{F}_T''} \backslash N_k$ , with  $\mathcal{F}_T' = \mathcal{F}_T' \left( -\sum_{i=1}^n \mu_i \cdot x_i \right)$ . In particular,  $\frac{k}{\bar{x}, \bar{\mu}}\mathcal{N}_y^\epsilon$  is a homogenous space for  $N_k$ .

Consider the case when  $\mathrm{pr}(\mu_i) + \mathrm{cond}_{x_i}(\varpi') \in \Lambda_{G_{\mathrm{ad}}}^{++}$ . The pull-back of  $\Psi_{\varpi'}^{\bar{x}, \bar{\mu}}$  to  $\frac{k}{\bar{x}, \bar{\mu}}\mathcal{N}_y^\epsilon$  is  $(N_k, \chi^k)$ -equivariant; we see that using the same reasoning as for  $\Psi_{\varpi'}$ . Since the isotropy subgroup  $N_{\mathrm{out}, k}^{\mathcal{F}_T''}$  is connected, any  $(N_k, \chi^k)$ -equivariant perverse sheaf is isomorphic to a direct sum of copies of this pull-back (up to the appropriate shift).

To prove the vanishing part of the statement it suffices to show the following: if  $\mathrm{pr}(\mu_i) + \mathrm{cond}_{x_i}(\varpi')$  is non-dominant for some  $i$  then the homomorphism

$$N_{\mathrm{out}, k}^{\mathcal{F}_T''} \hookrightarrow N_k \xrightarrow{\chi^k} \mathbb{G}_a$$

is non-trivial, for  $k$  sufficiently large. Clearly, it suffices to show that the composition  $N_{\mathrm{out}}^{\mathcal{F}_T''} \rightarrow N(\hat{\mathcal{K}}_y) \xrightarrow{\chi} \mathbb{G}_a$  is non-zero.

By definition, the above homomorphism  $N_{\mathrm{out}}^{\mathcal{F}_T''} \rightarrow \mathbb{G}_a$  is the composition:

$$N_{\mathrm{out}}^{\mathcal{F}_T''} \twoheadrightarrow \prod_{i \in \mathcal{J}} H^0(X - y, \mathcal{L}_{\mathcal{F}_T''}^{\check{\alpha}_i}) \rightarrow \prod_{i \in \mathcal{J}} H^0(\mathcal{D}_y^\times, \mathcal{L}_{\mathcal{F}_T''}^{\check{\alpha}_i}) \xrightarrow{\varpi'} \Omega_{\hat{\mathcal{K}}_y}^{\mathcal{J}} \xrightarrow{\mathrm{Res}} \mathbb{G}_a^{\mathcal{J}} \xrightarrow{\mathrm{sum}} \mathbb{G}_a.$$

Assume that for some  $j \in \{1, \dots, n\}$  and  $i \in \mathcal{J}$ ,  $\langle \mathrm{pr}(\mu_j) + \mathrm{cond}_{x_j}(\varpi'), \check{\alpha}_i \rangle < 0$ . Then there exists a section  $\gamma \in H^0(X - y, \mathcal{L}_{\mathcal{F}_T''}^{\check{\alpha}_i})$  with the following properties:

- $\gamma$  does not vanish under the composition

$$H^0(X - y, \mathcal{L}_{\mathcal{F}_T''}^{\check{\alpha}_i}) \rightarrow H^0(\mathcal{D}_{x_j}^\times, \mathcal{L}_{\mathcal{F}_T''}^{\check{\alpha}_i}) \xrightarrow{\varpi'_i} \Omega_{\hat{\mathcal{K}}_{x_j}} \xrightarrow{\mathrm{Res}} \mathbb{G}_a.$$

- $\gamma$  vanishes to a sufficiently high order at the points  $x_{j'}, j' \neq j$ , so that it goes to 0 under the composition

$$H^0(X - y, \mathcal{L}_{\mathcal{F}_T'}^{\check{\alpha}_i}) \rightarrow H^0(\mathcal{D}_{x_{j'}}^\times, \mathcal{L}_{\mathcal{F}_T'}^{\check{\alpha}_i}) \xrightarrow{\varpi'_i} \Omega_{\hat{\mathcal{K}}_{x_{j'}}} \xrightarrow{\text{Res}} \mathbb{G}_a.$$

But then by the residue formula, the image of  $\gamma$  under the composition

$$H^0(X - y, \mathcal{L}_{\mathcal{F}_T'}^{\check{\alpha}_i}) \rightarrow H^0(\mathcal{D}_y^\times, \mathcal{L}_{\mathcal{F}_T'}^{\check{\alpha}_i}) \xrightarrow{\varpi'_i} \Omega_{\hat{\mathcal{K}}_y} \xrightarrow{\text{Res}} \mathbb{G}_a$$

is non-zero. Therefore  $\chi(\gamma) \neq 0$ . This shows that  $\mathcal{S} = 0$  under the above assumption.

This completes the proof of Lemma 6.2.8 and hence of Lemma 6.2.3 and Lemma 6.2.4.

6.2.10. In order to complete the proof of Proposition 6.2.1(2), we need to show that the sheaf  $\Psi_{\varpi}^{\bar{x}, \bar{\nu}}!$  has a filtration whose consecutive quotients are perverse sheaves of the form  $\overline{\Psi}_{\varpi}^{\bar{x}, \bar{\mu}}$ . For simplicity, consider the case  $n = 1$ . We prove this statement by induction on  $\langle \nu, \check{\rho} \rangle$ .

If  $\nu = 0$ , then there are no relevant strata in the closure of  ${}_{x, \nu} \text{Bun}_N^{\mathcal{F}_T}$  and therefore  $\overline{\Psi}_{\varpi}^{x, 0} = \Psi_{\varpi}^{x, 0}!$ .

Now suppose that we have already proved the result for all dominant weights  $\nu$ , such that  $\langle \nu, \check{\rho} \rangle < N$ . Consider a dominant weight  $\nu$ , such that  $\langle \nu, \check{\rho} \rangle = N$ .

According to Proposition 6.2.1(1),  $\Psi_{\varpi}^{x, \nu}!$  is a perverse sheaf. Let  $\mathcal{K}$  denote the perverse sheaf that fits into the exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \Psi_{\varpi}^{x, \nu}! \rightarrow \overline{\Psi}_{\varpi}^{x, \nu} \rightarrow 0.$$

But then  $\mathcal{K}$  is non-zero only on the relevant strata  ${}_{x, \mu} \text{Bun}_N^{\mathcal{F}_T}$  with  $\langle \mu, \check{\rho} \rangle < N$ , and we know that the perverse cohomology groups of its restrictions to those strata are direct sums of copies of  $\Psi_{\varpi}^{x, \mu}$ . Hence, on the level of Grothendieck groups, we can write  $\mathcal{K}$  as a combination of the sheaves  $\Psi_{\varpi}^{x, \mu}!$  with  $\langle \mu, \check{\rho} \rangle < N$ . By our inductive assumption, the sheaves  $\Psi_{\varpi}^{x, \mu}!$  are consecutive extensions of perverse sheaves of the form  $\overline{\Psi}_{\varpi}^{x, \mu'}$ , and hence so is  $\mathcal{K}$  and, finally, so is  $\Psi_{\varpi}^{x, \nu}!$ .

Thus, Proposition 6.2.1 is proved.

6.2.11. *Proof of Theorem 3(2).* Let us show that the map

$$(6.6) \quad \Psi_{\varpi}^{\bar{x}, \bar{\nu}}! \rightarrow \overline{\Psi}_{\varpi}^{\bar{x}, \bar{\nu}}$$

is an isomorphism. The isomorphism  $\Psi_{\varpi}^{\bar{x}, \bar{\nu}} * \simeq \overline{\Psi}_{\varpi}^{\bar{x}, \bar{\nu}}$  will then also follow by Verdier duality.

Let  $\mathcal{K}$  denote the kernel of the map (6.6). According to Proposition 6.2.1,  $\mathcal{K}$  is an object of  $\mathcal{W}_{\varpi}^{\bar{x}}$  and by Theorem 3(1), we have:

$$\Psi_{\varpi}^{\bar{x}, \bar{\nu}}! \simeq \overline{\Psi}_{\varpi}^{\bar{x}, \bar{\nu}} \oplus \mathcal{K}.$$

However, the restriction of  $\Psi_{\varpi}^{\bar{x}, \bar{\nu}}!$  to the complement of  ${}_{\bar{x}, \bar{\nu}} \text{Bun}_N^{\mathcal{F}_T}$  in  ${}_{\bar{x}, \bar{\nu}} \overline{\text{Bun}}_N^{\mathcal{F}_T}$  is zero, by definition. Therefore the same is true for  $\mathcal{K}$ .

But then  $\mathcal{K} = 0$ , since by construction  $\mathcal{K}$  is supported on  ${}_{\bar{x}, \bar{\nu}} \overline{\text{Bun}}_N^{\mathcal{F}_T} - {}_{\bar{x}, \bar{\nu}} \text{Bun}_N^{\mathcal{F}_T}$ . Hence (6.6) is an isomorphism.

## 7. COMPUTATION OF THE HECKE ACTION

In this section we collect several facts concerning the geometry of  $N(\hat{\mathcal{K}}_x)$ -orbits on  $\text{Gr}$  and prove Theorem 4.

7.1. Semi-infinite orbits on  $\text{Gr}$ .

7.1.1. Recall the ind-scheme  $\text{Gr}$ . Informally, one defines the locally closed ind-subscheme  $S^\nu \subset \text{Gr}$  as the  $N(\hat{\mathcal{K}}_x)$ -orbit of the point  $\nu(t) \in \text{Gr}$ . Here is a precise scheme-theoretic definition.

The chosen Borel subgroup  $B \subset G$  defines for each  $\check{\lambda} \in \check{\Lambda}^{++}$  a line subbundle  $\mathcal{O} \subset \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$  over  $\mathcal{D}_x$ . We say that a point  $(\mathcal{F}_G, \beta) \in \text{Gr}$  belongs to  $\overline{S}^\nu$ , if for every  $\check{\lambda}$  the map  $\mathcal{O} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}(\infty \cdot x)$  obtained using the data of  $\beta$  factors as

$$(7.1) \quad \mathcal{O} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}(\langle \nu, \check{\lambda} \rangle \cdot x) \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}(\infty \cdot x).$$

It follows from the definition that there is a morphism  $\overline{S}^\nu \rightarrow_{x, \nu} \overline{\text{Bun}}_N^{\mathcal{F}_G^0}$ , which corresponds to forgetting  $\beta$ , but keeping the maps  $\mathcal{O} \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}(\langle \nu, \check{\lambda} \rangle \cdot x)$  induced by  $\beta$ . One can show that this morphism is formally smooth.

It is clear that  $\overline{S}^\nu$  is a closed ind-subscheme of  $\text{Gr}$  and  $\overline{S}^{\nu'} \subset \overline{S}^\nu$  if and only if  $\nu \geq \nu'$ . We define the locally closed ind-subscheme  $S^\nu \subset \text{Gr}$  as follows: we say that  $(\mathcal{F}_G, \beta) \in \text{Gr}$  belongs to  $S^\nu$ , if for every  $\check{\lambda}$  the map  $\mathcal{O} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}(\infty \cdot x)$  factors as (7.1) and the embedding  $\mathcal{O} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}(\langle \nu, \check{\lambda} \rangle \cdot x)$  is maximal.

The group  $N(\hat{\mathcal{K}}_x)$  acts naturally on  $\text{Gr}$ . It is easy to see that each subscheme  $S^\nu$  is  $N(\hat{\mathcal{K}}_x)$ -stable. Moreover, one can show that the  $N(\hat{\mathcal{K}}_x)$ -action on  $S^\nu$  is transitive in the sense that  $S^\nu = \bigcup_k (S^\nu)^k$ , where each  $(S^\nu)^k$  is a homogeneous space for  $N^k$ .

7.1.2. *Remark.* It is instructive to compute explicitly  $R$ -points of the scheme  $\overline{S}^\nu$ , when  $G = SL_2$  and  $R$  is an Artinian local ring. In this case, the set  $\overline{S}^n(R) \subset \text{Gr}(R)$  consists of the cosets of the form

$$\begin{pmatrix} 1 & u(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{n-m} p_m(t^{-1})^{-1} & 0 \\ 0 & t^{-n+m} p_m(t^{-1}) \end{pmatrix} \cdot SL_2(R[[t]]), \quad m \geq 0$$

where  $u(t) \in R((t))$ , and  $p_m(t^{-1})$  is an invertible element of  $R[t^{-1}]$  of degree  $m$ . This means that (up to a scalar in  $R^\times$ , which can be absorbed into  $SL_2(R[[t]])$ )

$$p_m(t^{-1}) = 1 + r_1 t^{-1} + \dots + r_m t^{-m},$$

where each coefficient  $r_i$  is a nilpotent element of  $R$ .

Thus, we see that the set of  $R$ -points of  $\overline{S}^n$  coincides with the set of  $R$ -points of a union of strata which are fibrations over  $\text{Sym}^m \mathcal{D}_x$  with fibers  $S^{n-m}$ , where  $m \geq 0$ .<sup>4</sup> These fibrations are similar to those described in Corollary 2.2.9: the formal disc  $\mathcal{D}_x$  here plays the role of the curve  $X$ .

<sup>4</sup>Here  $\text{Sym}^m \mathcal{D}_x$  is a formal scheme  $\text{Spf}(\overline{\mathbb{F}}_q[[t_1, \dots, t_m]]^{\Sigma^m})$ , where  $\Sigma^m$  is the symmetric group.

This is analogous to the description of  $\overline{S}^n$  obtained in [23] in the case when  $\hat{\mathcal{O}}_x$  is replaced by the ring of analytic functions on the unit disc – then  $\mathcal{D}_x$  is replaced by the unit disc.

7.1.3. The following proposition, due to [18], will play an important role in the proof of Theorem 4.

**Proposition.**  $\dim(\mathrm{Gr}^\lambda \cap S^\nu) \leq \langle \lambda + \nu, \check{\rho} \rangle$ .

It is also known that for  $\nu = \lambda$ ,  $\mathrm{Gr}^\lambda \cap S^\nu$  is open and dense in  $\mathrm{Gr}^\lambda$  and for  $\nu = w_0(\lambda)$ ,  $\mathrm{Gr}^\lambda \cap S^\nu$  is a point-scheme.

7.1.4. *Admissible characters.* Let  $\eta$  be a coweight of  $G_{\mathrm{ad}}$  and let us choose isomorphisms of  $\hat{\mathcal{O}}_x$ -modules

$$(7.2) \quad \hat{\mathcal{O}}_x(\langle \eta, \check{\alpha}_i \rangle \cdot x) \simeq \Omega|_{\mathcal{D}_x}$$

for each  $i \in \mathcal{I}$ .

As in Sect. 6.2.6, the above data define a homomorphism  $\chi_\eta : N(\hat{\mathcal{K}}_x) \rightarrow \mathbb{G}_a$ . We call such a character admissible of conductor  $\eta$ . It is clear that the group  $T(\hat{\mathcal{O}}_x)(\overline{\mathbb{F}}_q)$  acts transitively on the set admissible characters with a given conductor.

In order to simplify notation, we will write  $\chi_\nu$  for  $\chi_{\mathrm{pr}(\nu)}$  for any  $\nu \in \Lambda$ .

7.1.5. **Lemma.** *Let  $\nu$  and  $\mu$  be two elements of  $\Lambda$ . Then for a given admissible character  $\chi_\mu$  of conductor  $\mathrm{pr}(\mu)$  there exists a  $(N(\hat{\mathcal{K}}_x), \chi_\mu)$ -equivariant function  $\chi'_\mu : S^\nu \rightarrow \mathbb{G}_a$  if and only if  $\mu + \nu \in \Lambda^{++}$ . In the latter case this function is unique up to an additive constant.*

7.1.6. When  $\mu + \nu \in \Lambda^{++}$ , the function  $\chi'_\mu : S^\nu \rightarrow \mathbb{G}_a$  can be defined by the formula

$$(7.3) \quad \chi'_\mu(n \cdot \nu(t)) = \chi(\mu(t) \cdot n \cdot \mu^{-1}(t)), \quad n \in N(\hat{\mathcal{K}}).$$

The next statement, which is proved in Sect. 7.3, is a key ingredient in the proof of Theorem 4.

7.1.7. **Proposition.** *Let  $\chi'_{-\nu} : S^\nu \rightarrow \mathbb{G}_a$  be an  $(N(\hat{\mathcal{K}}_x), \chi_{-\nu})$ -equivariant function. Assume that  $\lambda \in \Lambda$  is a dominant coweight and  $\nu \neq w_0(\lambda)$ . Then the restriction of  $\chi'_{-\nu}$  to any irreducible component of  $\mathrm{Gr}^\lambda \cap S^\nu$  of dimension  $\langle \lambda + \nu, \check{\rho} \rangle$  is a dominant map onto  $\mathbb{G}_a$ .*

7.1.8. **Corollary.** *The cohomology  $H_c^{(\lambda+\nu, 2\check{\rho})}(\mathrm{Gr}^\lambda \cap S^\nu, \chi'_{-\nu}|_{\mathrm{Gr}^\lambda \cap S^\nu}^*(\mathcal{I}_\psi))$  vanishes unless  $\nu = w_0(\lambda)$ .*

7.1.9. *Proof.* Without loss of generality we can assume that  $\chi_{-\nu}^\nu(\nu(t)) = 0 \in \mathbb{G}_a$ .

Denote by  $K$  the complex on  $\mathbb{G}_a$ , which is the  $!$ -direct image of the constant sheaf on  $\mathrm{Gr}^\lambda \cap S^\nu$  with respect to the map  $\chi_{-\nu}^\nu : \mathrm{Gr}^\lambda \cap S^\nu \rightarrow \mathbb{G}_a$ .

Consider the  $T$ -action on  $\mathbb{G}_a$  given by the character  $2\check{\rho} : T \rightarrow \mathbb{G}_m$ . It is easy to see that  $K$  is  $T$ -equivariant with respect to this action. In particular, it is monodromic with respect to the standard  $\mathbb{G}_m$ -action on  $\mathbb{G}_a$ .

Proposition 7.1.7 and Proposition 7.1.3 imply that our complex lives in the (perverse) cohomological degrees strictly less than  $\langle \lambda + \nu, 2\check{\rho} \rangle$ . Corollary 7.1.8 is therefore equivalent to saying that  $H_c^1(\mathbb{G}_a, \tilde{K} \otimes \mathcal{I}_\psi) = 0$ , where  $\tilde{K}$  is the  $(\langle \lambda + \nu, 2\check{\rho} \rangle - 1)$ -th perverse cohomology sheaf of  $K$ . But this follows from:

7.1.10. **Lemma.** *Let  $\mathcal{S}$  be a  $\mathbb{G}_m$ -monodromic perverse sheaf on  $\mathbb{A}^1$ . Then we have:  $H_c^1(\mathbb{A}^1, \mathcal{S} \otimes \mathcal{I}_\psi) = 0$ .*

## 7.2. Proof of Theorem 4.

7.2.1. For notational convenience, we replace  $\lambda$  by  $-w_0(\lambda)$  and prove the statement:

$$\overline{\Psi}_{\varpi}^{x,0} \star \mathcal{A}_{-w_0(\lambda)} = \overline{\Psi}_{\varpi}^{x,-w_0(\lambda)}.$$

Thus, we need to establish an isomorphism

$$(7.4) \quad {}'h_!^{\leftarrow}((\overline{\Psi}_{\varpi}^{x,0} \boxtimes \mathcal{A}_{-w_0(\lambda)})^r) \simeq \overline{\Psi}_{\varpi}^{x,-w_0(\lambda)}.$$

To simplify notation, from now on we will suppress the upper index  $r$  in this formula.

Denote by  $\tilde{K}^\nu$  (resp.,  $K^\nu$ ) the  $*$ -restriction of the LHS of (7.4) to the stratum  ${}_{x,\nu}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T}$  (resp.,  ${}_{x,\nu}\mathrm{Bun}_N^{\mathcal{F}_T}$ ). Since the both sides of (7.4) are Verdier self-dual (up to replacing  $\psi$  by  $\psi^{-1}$ ) it suffices to prove the following

**Claim 1.** (1) *The complex  $\tilde{K}^\nu$  lives in the (perverse) cohomological degrees  $\leq 0$ .*

(2) *The  $*$ -restriction of  $\tilde{K}^\nu$  to the closed substack*

$${}_{x,\nu}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T} - {}_{x,\nu}\mathrm{Bun}_N^{\mathcal{F}_T} \subset {}_{x,\nu}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T}$$

*lives in strictly negative (perverse) cohomological degrees.*

(3) *The 0-th (perverse) cohomology of  $K^\nu$  vanishes unless  $\nu = -w_0(\lambda)$  and in the latter case it can be identified with  $\Psi_{\varpi}^{x,-w_0(\lambda)}$ .*

7.2.2. For each pair  $\nu, \nu' \in \Lambda$  consider the following locally closed substacks of  $Z$ :

$$\begin{aligned} \tilde{Z}^{\nu,?} &:= {}'h^{\leftarrow-1}({}_{x,\nu}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T}), \quad Z^{\nu,?} := {}'h^{\leftarrow-1}({}_{x,\nu}\mathrm{Bun}_N^{\mathcal{F}_T}) \\ \tilde{Z}^{?,\nu'} &:= {}'h^{\rightarrow-1}({}_{x,\nu'}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T}), \quad Z^{?,\nu'} := {}'h^{\rightarrow-1}({}_{x,\nu'}\mathrm{Bun}_N^{\mathcal{F}_T}) \\ \tilde{Z}^{\nu,\nu'} &:= \tilde{Z}^{\nu,?} \cap \tilde{Z}^{?,\nu'}, \quad Z^{\nu,\nu'} := Z^{\nu,?} \cap Z^{?,\nu'}. \end{aligned}$$

For  $\mu \in \Lambda^{++}$ , denote by  $\mathcal{H}_x^\mu$  the locally closed substack  $\mathcal{G}_x \times_{G(\check{\mathcal{O}}_x)} \mathrm{Gr}^\mu$  of  $\mathcal{H}_x$  (the projection onto  $\mathrm{Bun}_G$  that we are using here is  $h^{\leftarrow}$ ), and set

$$\begin{aligned}\tilde{Z}^{\nu,?,\mu} &:= \tilde{Z}^{\nu,?} \cap {}'\overline{\mathfrak{p}}^{-1}(\mathcal{H}_x^\mu), \quad \tilde{Z}^{?,\nu',\mu} := \tilde{Z}^{?,\nu'} \cap {}'\overline{\mathfrak{p}}^{-1}(\mathcal{H}_x^\mu) \\ \tilde{Z}^{\nu,\nu',\mu} &:= \tilde{Z}^{\nu,\nu'} \cap {}'\overline{\mathfrak{p}}^{-1}(\mathcal{H}_x^\mu), \quad Z^{\nu,\nu',\mu} := Z^{\nu,\nu'} \cap {}'\overline{\mathfrak{p}}^{-1}(\mathcal{H}_x^\mu)\end{aligned}$$

Let us denote by  $\tilde{K}^{\nu,\nu',\mu}$  (resp., by  $K^{\nu,\nu',\mu}$ ) the  $!$ -direct image under

$${}'h^\leftarrow : \tilde{Z}^{\nu,\nu',\mu} \rightarrow {}_{x,\nu}\widetilde{\text{Bun}}_N^{\mathcal{F}_T} \quad (\text{resp., } Z^{\nu,\nu',\mu} \rightarrow {}_{x,\nu}\text{Bun}_N^{\mathcal{F}_T})$$

of the  $*$ -restriction of  $\overline{\Psi}_\varpi^{x,0} \boxtimes \mathcal{A}_{-w_0(\lambda)}$  to  $\tilde{Z}^{\nu,\nu',\mu}$  (resp., to  $Z^{\nu,\nu',\mu}$ ).

Using a standard spectral sequence, one can derive Claim 1 from the following

**Claim 2.** (1) *The complex  $\tilde{K}^{\nu,\nu',\mu}$  lives in cohomological degrees  $\leq 0$  and the inequality is strict unless  $\nu' = 0$  and  $\mu = \lambda$ .*

(2) *The  $*$ -restriction of  $\tilde{K}^{\nu,0,\lambda}$  to the closed substack*

$${}_{x,\nu}\widetilde{\text{Bun}}_N^{\mathcal{F}_T} - {}_{x,\nu}\text{Bun}_N^{\mathcal{F}_T} \subset {}_{x,\nu}\widetilde{\text{Bun}}_N^{\mathcal{F}_T}$$

*lives in strictly negative cohomological degrees.*

(3) *The 0-th cohomology of  $K^{\nu,0,\lambda}$  vanishes unless  $\nu = -w_0(\lambda)$ .*

(4) *The 0-th cohomology of  $K^{-w_0(\lambda),0,\lambda}$  can be identified with  $\Psi_\varpi^{x,-w_0(\lambda)}$ .*

The proof of Claim 2 will be obtained by analyzing the fibers of the projection  ${}'h^\leftarrow : \tilde{Z}^{\nu,\nu',\mu} \rightarrow {}_{x,\nu}\widetilde{\text{Bun}}_N^{\mathcal{F}_T}$ .

7.2.3. Recall that by choosing a trivialization  $\epsilon : \mathcal{F}_T \rightarrow \mathcal{F}_T^0|_{\mathcal{D}_x}$ , we obtain canonical  $N(\hat{\mathcal{O}}_x)$ -torsors  $\tilde{\mathcal{N}}_x^\epsilon$  and  $\mathcal{N}_x^\epsilon$  over  ${}_{x,0}\widetilde{\text{Bun}}_N^{\mathcal{F}_T}$  and  $\text{Bun}_N^{\mathcal{F}_T}$ , respectively.

Independently, for every  $\nu \in \Lambda$  and  $\mathcal{F}'_T := \mathcal{F}_T(-\nu \cdot x)$  we fix a trivialization  $\epsilon_\nu : \mathcal{F}'_T \rightarrow \mathcal{F}_T^0|_{\mathcal{D}_x}$ . Using the isomorphism  ${}_{x,\nu}\widetilde{\text{Bun}}_N^{\mathcal{F}'_T} \simeq \widetilde{\text{Bun}}_N^{\mathcal{F}'_T}$  we obtain  $N(\hat{\mathcal{O}}_x)$ -torsors  ${}_\nu\tilde{\mathcal{N}}_x^{\epsilon_\nu}$  and  ${}_\nu\mathcal{N}_x^{\epsilon_\nu}$  over  ${}_{x,\nu}\widetilde{\text{Bun}}_N^{\mathcal{F}_T}$  and  ${}_{x,\nu}\text{Bun}_N^{\mathcal{F}_T}$ , respectively.

By definition, the projection  ${}'h^\leftarrow$  identifies  $\tilde{Z}^{\nu,?}$  with the fibration

$${}_\nu\tilde{\mathcal{N}}_x^{\epsilon_\nu} \times_{N(\hat{\mathcal{O}}_x)} \text{Gr} \rightarrow {}_{x,\nu}\widetilde{\text{Bun}}_N^{\mathcal{F}_T},$$

and similarly, the projection  ${}'h^\rightarrow$  realizes  $\tilde{Z}^{?,\nu'}$  as a fibration

$${}_{\nu'}\tilde{\mathcal{N}}_x^{\epsilon_{\nu'}} \times_{N(\hat{\mathcal{O}}_x)} \text{Gr} \rightarrow {}_{x,\nu'}\widetilde{\text{Bun}}_N^{\mathcal{F}_T}.$$

Recalling the definition of the subscheme  $S^\nu \subset \text{Gr}$  from Sect. 7.1.1, we obtain the following lemma:

7.2.4. **Lemma.**

(1) *The stacks  $\tilde{Z}^{\nu,\nu'}$  and  $\tilde{Z}^{\nu,?,\mu}$ , when viewed as substacks of  $\tilde{Z}^{\nu,?}$ , can be identified with the fibrations*

$${}_\nu\tilde{\mathcal{N}}_x^{\epsilon_\nu} \times_{N(\hat{\mathcal{O}}_x)} S^{\nu'-\nu} \xrightarrow{{}'h^\leftarrow} {}_{x,\nu}\widetilde{\text{Bun}}_N^{\mathcal{F}_T} \quad \text{and} \quad {}_{\nu'}\tilde{\mathcal{N}}_x^{\epsilon_{\nu'}} \times_{N(\hat{\mathcal{O}}_x)} \text{Gr}^\mu \xrightarrow{{}'h^\leftarrow} {}_{x,\nu}\widetilde{\text{Bun}}_N^{\mathcal{F}_T}, \quad \text{respectively.}$$

(2) The stacks  $\widetilde{Z}^{\nu,\nu'}$  and  $\widetilde{Z}^{?,\nu',\mu}$ , when viewed as substacks of  $\widetilde{Z}^{?,\nu'}$ , can be identified with the fibrations

$${}_{\nu'}\widetilde{\mathcal{N}}_x^{\epsilon_{\nu'}} \times_{N(\hat{\mathcal{O}}_x)} S^{\nu-\nu'} \xrightarrow{'h^{\rightarrow}} {}_{x,\nu'}\widetilde{\text{Bun}}_N^{\mathcal{F}_T} \text{ and } {}_{\nu'}\widetilde{\mathcal{N}}_x^{\epsilon_{\nu'}} \times_{N(\hat{\mathcal{O}}_x)} \text{Gr}^{-w_0(\mu)} \xrightarrow{'h^{\rightarrow}} {}_{x,\nu'}\widetilde{\text{Bun}}_N^{\mathcal{F}_T}, \text{ respectively.}$$

(3) The substack  $Z^{\nu,\nu'}$  coincides with the intersection  $\widetilde{Z}^{\nu,?} \cap Z^{?,\nu'}$ .

7.2.5. Now, we are ready to prove the first two statements of Claim 2.

Indeed, from Lemma 7.2.4 we conclude that the  $*$ -restriction of  $\widetilde{\Psi}_{\varpi}^{x,0} \boxtimes \mathcal{A}_{-w_0(\lambda)}$  to  $\widetilde{Z}^{?,\nu',\mu}$  is the twisted external product of complexes:

$$(7.5) \quad (\widetilde{\Psi}_{\varpi}^{x,0} |_{{}_{x,\nu'}\widetilde{\text{Bun}}_N^{\mathcal{F}_T}}) \boxtimes (\mathcal{A}_{-w_0(\lambda)} |_{\text{Gr}^{-w_0(\mu)}}).$$

By definition, it lives in the cohomological degrees  $\leq 0$  and the inequality is strict unless  $\mu = -w_0(\lambda)$  and  $\nu' = 0$ . (In fact, we know from Proposition 6.2.1 that the above complex is 0 unless  $\nu' = 0$ , but we will not need this fact here.)

Now since  $\mathcal{A}_{-w_0(\lambda)}|_{\text{Gr}^{-w_0(\mu)}}$  is constant, it follows from Lemma 7.2.4(2) that its further  $*$ -restriction to  $\widetilde{Z}^{\nu,\nu',\mu}$  lives in (perverse) cohomological degrees

$$\leq -\text{codim}(S^{\nu-\nu'} \cap \text{Gr}^{-w_0(\mu)}, \text{Gr}^{-w_0(\mu)}) \leq -\langle -w_0(\mu) - \nu + \nu', \check{\rho} \rangle = -\langle \mu - \nu + \nu', \check{\rho} \rangle,$$

according to Proposition 7.1.3.

However, we obtain from Lemma 7.2.4(1) and Proposition 7.1.3 that the fibers of the projection  $'h^{\leftarrow} : \widetilde{Z}^{\nu,\nu',\mu} \rightarrow {}_{x,\nu'}\widetilde{\text{Bun}}_N^{\mathcal{F}_T}$  are of dimension  $\leq \langle \mu - \nu + \nu', \check{\rho} \rangle$ .

This proves Claim 2(1).

Claim 2(2) follows from Lemma 7.2.4(3) combined with the above dimension estimates and the fact that the  $*$ -restriction of  $\widetilde{\Psi}_{\varpi}^{x,0}$  to  ${}_{x,0}\widetilde{\text{Bun}}_N^{\mathcal{F}_T} - {}_{x,0}\text{Bun}_N^{\mathcal{F}_T}$  lives in strictly negative cohomological degrees.

Thus, it remains to study  $K^{\nu,0,\lambda}$ .

7.2.6. Let  $\overline{N(K_x)}^{\epsilon_{\nu}}$  denote the ind-group scheme over  ${}_{x,\nu}\text{Bun}_N^{\mathcal{F}_T}$ , obtained as a  ${}_{\nu}\mathcal{N}_x^{\epsilon_{\nu}}$ -twist of  $N(\hat{\mathcal{K}}_x)$  with respect to the adjoint action of  $N(\hat{\mathcal{O}}_x)$  on  $N(\hat{\mathcal{K}}_x)$ . The description of  $Z^{\nu,\nu'}$  as a stack fibered over  ${}_{x,\nu}\text{Bun}_N^{\mathcal{F}_T}$  in Lemma 7.2.4(1) implies that  $Z^{\nu,\nu'}$  carries a canonical  $\overline{N(K_x)}^{\epsilon_{\nu}}$ -action.

Recall now from Sect. 7.1.4 that the data of  $\epsilon_{\nu}$  and  $\varpi$  give rise to an admissible character  $\chi_{\nu} : N(\hat{\mathcal{K}}_x) \rightarrow \mathbb{G}_a$ , of conductor  $\text{pr}(\nu)$ . Hence we also obtain a character on the above ind-group scheme  $\overline{N(K_x)}^{\epsilon_{\nu}}$ , which we denote by  $\overline{\chi}_{\nu}$ .

For dominant coweights  $\nu$  and  $\nu'$ , the  $(N(\hat{\mathcal{K}}_x), \chi_{\nu})$ -equivariant function  $\chi_{\nu}^{\nu'-\nu} : S^{\nu'-\nu} \rightarrow \mathbb{G}_a$  gives rise to a  $(\overline{N(K_x)}^{\epsilon_{\nu}}, \overline{\chi}_{\nu})$ -equivariant function  $\overline{\chi}_{\nu}^{\nu'-\nu} : Z^{\nu,\nu'} \rightarrow \mathbb{G}_a$ .

The following result is obtained directly from the definitions:

7.2.7. **Lemma.** Assume that  $\nu' \in \Lambda$  is dominant. Then

(1) The function

$$(\text{ev}_{\varpi}^{x,\nu'} \circ 'h^{\rightarrow}) : Z^{\nu,\nu'} \rightarrow \mathbb{G}_a$$

is  $(\overline{N(K_x)}^{\epsilon_{\nu}}, \overline{\chi}_{\nu})$ -equivariant.

(2) If  $\nu$  is also dominant, then the above function coincides with the composition

$$Z^{\nu, \nu'} \xrightarrow{\bar{\chi}_\nu^{\nu' - \nu} \times h'^{-1}} \mathbb{G}_a \times_{x, \nu} \mathrm{Bun}_N^{\mathcal{F}_T} \xrightarrow{\mathrm{id} \times \mathrm{ev}_\omega^{\nu, x}} \mathbb{G}_a \times \mathbb{G}_a \xrightarrow{\mathrm{sum}} \mathbb{G}_a$$

for some  $\chi_\nu^{\nu' - \nu}$ .

7.2.8. Assume that  $\nu$  is different from  $-w_0(\lambda)$ . Lemma 7.2.4(1) implies that after a smooth localization  $U \rightarrow_{x, \nu} \mathrm{Bun}_N^{\mathcal{F}_T}$  (e.g., we can take  $U = {}^k\nu\mathcal{N}_x^{\epsilon_\nu}$ , for large enough  $k$ ), the fibration  $'h^{\leftarrow} : Z^{\nu, 0, \lambda} \rightarrow_{x, \nu} \mathrm{Bun}_N^{\mathcal{F}_T}$  becomes a direct product  $U \times (\mathrm{Gr}^\lambda \cap S^{-\nu})$ .

Moreover, by Lemma 7.2.7(1), the complex  $(\bar{\Psi}_\omega^{x, 0} \boxtimes \tilde{\mathcal{A}}_{-w_0(\lambda)})|_{Z^{\nu, 0, \lambda}}$  becomes the external product

$$\mathcal{E} \boxtimes \chi_\nu^{-\nu*}(\mathcal{J}_\psi)[\langle \lambda - \nu, 2\check{\rho} \rangle],$$

where  $\mathcal{E}$  is a locally constant perverse sheaf on the base  $U$ . Therefore Claim 2(3) follows from Corollary 7.1.8.

Finally, for  $\nu = -w_0(\lambda)$  the intersection  $S^{w_0(\lambda)} \cap \mathrm{Gr}^\lambda$  is a point-scheme (cf. Sect. 7.1.1) and therefore Claim 2(4) follows from Lemma 7.2.7(2) applied to the case  $\nu' = 0$ .

In order to finish the proof of Claim 2 (and hence of Theorem 4), it remains to prove Proposition 7.1.7.

**7.3. Proof of Proposition 7.1.7.** In order to prove the proposition, it suffices to study the function  $\chi_{-\nu}^\nu$  on the set of  $\bar{\mathbb{F}}_q$ -points of  $\mathrm{Gr}^\lambda \cap S^\nu$ .

7.3.1. *Computation in rank 1.* First we will consider the case when  $G$  has semi-simple rank 1. We can identify each intersection of an  $N(\hat{\mathcal{K}}_x)$ -orbit and an  $G(\hat{\mathcal{O}}_x)$ -orbit in  $\mathrm{Gr}_G$  with an appropriate intersection of an  $N(\hat{\mathcal{K}}_x)$ -orbit and an  $G_{\mathrm{ad}}(\hat{\mathcal{O}}_x)$ -orbit in  $\mathrm{Gr}_{G_{\mathrm{ad}}}$ , where  $G_{\mathrm{ad}} = G/Z(G)$ . Moreover, the corresponding functions  $\chi_{-\nu}^\nu$  coincide under this identification. Therefore without loss of generality we can replace  $G$  by the corresponding adjoint group, so it suffices to treat the case of the group  $PGL(2)$ . In this case we prove the statement of Proposition 7.1.7 by an explicit computation as follows.

Let us identify  $\Lambda$  with  $\mathbb{Z}$ . The intersection  $\overline{\mathrm{Gr}}^m \cap S^n$  is empty unless  $m$  and  $n$  have the same parity and  $|n| \leq m$ . When this is the case,  $\overline{\mathrm{Gr}}^m \cap S^n$  is isomorphic to  $\mathbb{A}^{(n+m)/2}$ :

$$(7.6) \quad \overline{\mathrm{Gr}}^m \cap S^n = \left\{ \begin{pmatrix} 1 & \sum_{i=(n-m)/2}^{n-1} a_i t^i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \mid a_i \in \bar{\mathbb{F}}_q \right\}.$$

In particular,  $\overline{\mathrm{Gr}}^m \cap S^n$  is always irreducible.

Hence in order to prove Proposition 7.1.7 it is enough to show that on the level of  $\bar{\mathbb{F}}_q$ -points, the function

$$\chi_{-n}^n : (\overline{\mathrm{Gr}}^m \cap S^n) \rightarrow \bar{\mathbb{F}}_q$$

is non-constant whenever  $\overline{\mathrm{Gr}}^m \cap S^n$  has positive dimension.

Using the  $a_i$  in (7.6) as coordinates on  $\overline{\mathrm{Gr}}^m \cap S^n$ , the function  $\chi_{-n}^n$  is given by  $\psi(a_{n-1})$ . Hence the restriction of  $\chi_{-n}^n$  to  $\overline{\mathrm{Gr}}^m \cap S^n$  is indeed non-constant if the latter has positive dimension.

**7.3.2. Grassmannians associated to the parabolic subgroups of  $G$ .** The proof for general  $G$  will be obtained by reduction from  $G$  to its minimal Levi subgroups. In what follows, the subscript “ $M$ ” (e.g. in  $\text{Gr}_M$ ,  $S_M^\nu$ , etc.) will denote the corresponding object for a Levi subgroup  $M$  of  $G$ .

Let  $P$  be a parabolic subgroup of  $G$ ,  $N_P$  its unipotent radical, and  $M$  its Levi subgroup. Denote by  $\mathcal{J}_M \subset \mathcal{J}$  the corresponding subset of  $\mathcal{J}$ . Let  $\Lambda_{G,P}$  be the quotient of  $\Lambda = \Lambda_M$  by the sublattice spanned by  $\alpha_i, i \in \mathcal{J}_M$ . Then  $\Lambda_{G,P}$  can be identified with the set of connected components of the Grassmannian  $\text{Gr}_M = M(\hat{\mathcal{K}}_x)/M(\hat{\mathcal{O}}_x)$ . We denote by  $\text{Gr}_M^{(\theta)}$  the component of  $\text{Gr}_M$  corresponding to  $\theta \in \Lambda_{G,P}$ . Note that  $\text{Gr}_M^\mu \subset \text{Gr}_M^{(\theta)}$  if and only if  $\mu$  belongs to the coset of  $\theta$  in  $\Lambda$ .

Let  $P_0(\hat{\mathcal{K}}_x)$  be a ind-subgroup of  $P(\hat{\mathcal{K}}_x)$ , defined as the inverse image of the subgroup  $M/[M, M](\hat{\mathcal{O}}_x)$  under the natural projection  $P(\hat{\mathcal{K}}_x) \rightarrow M(\hat{\mathcal{K}}_x) \rightarrow M/[M, M](\hat{\mathcal{K}}_x)$ . As in Sect. 7.1.1, to an element  $\theta \in \Lambda_{G,P}$  one can attach a locally closed ind-subscheme  $S_P^\theta$  in  $\text{Gr}$ , on which  $P_0(\hat{\mathcal{K}}_x)$  acts transitively (at the level of points,  $S_P^\theta$  is the  $P_0(\hat{\mathcal{K}}_x)$ -orbit of  $\theta(t)$  in  $\text{Gr}$ ).

In the same way as in Sect. 5.1.1 we can define the affine Grassmannian  $\text{Gr}_P$  (note that the definition given in Sect. 5.1.1 works for any algebraic group). The set of connected components of  $\text{Gr}_P$  coincides with that of  $\text{Gr}_M$ . Now  $S_P^\theta$  is nothing but the reduced scheme of the corresponding connected component of  $\text{Gr}_P$ . Thus, the natural projection  $\text{Gr}_P \rightarrow \text{Gr}_M$  gives rise to a map  $p^{(\theta)} : S_P^\theta \rightarrow \text{Gr}_M^{(\theta)}$ .

The following statement is straightforward.

**7.3.3. Lemma.** *For  $\nu \in \Lambda$  let  $\theta$  be its image in  $\Lambda_{G,P}$ . Then  $S^\nu \subset S_P^\theta$  and  $p^{(\theta)}(S^\nu) \subset S_M^\nu$ .*

Denote by  $p^\nu$  the restriction of  $p^{(\theta)}$  to  $S^\nu$ .

**7.3.4.** For  $i \in \mathcal{J}$ , denote by  $P_i$  the corresponding minimal parabolic subgroup of  $G$  and by  $M_i$  its Levi subgroup. Then we have a morphism  $p_i^\nu : S^\nu \rightarrow S_{M_i}^\nu$ . Note that  $S_{M_i}^\nu$  is always irreducible as was shown in Sect. 7.3.1 by an explicit calculation.

Now let  $\lambda \in \Lambda^{++}$  and  $\nu \in \Lambda$  be such that  $\nu \neq w_0(\lambda)$ , i.e., so that  $\text{Gr}^\lambda \cap S^\nu$  has positive dimension. Let  $K$  be an irreducible component of dimension  $\langle \lambda + \mu, \check{\rho} \rangle$  of  $\text{Gr}^\lambda \cap S^\nu$ .

The following result is proved in [3]:

**7.3.5. Proposition.** *For  $\lambda, \nu$  and  $K$  as above, there exist  $i \in \mathcal{J}$  and  $\mu \in \Lambda_{M_i}^{++}$  such that an open dense subset of  $K$  projects dominantly onto  $\text{Gr}_{M_i}^\mu \cap S_{M_i}^\nu$  under the map  $p_i^\nu$  and  $\nu \neq w_{0, M_i}(\mu)$ .*

(In the above formula  $w_{0, M_i}$  is the “longest” element of the Weyl group of  $M_i$ , i.e. the  $i$ -th simple reflection.)

**7.3.6.** Recall from Sects. 6.2.6 and 7.1.4 that the character  $\chi_{-\nu} : N(\hat{\mathcal{K}}_x) \rightarrow \mathbb{G}_a$  is by definition a sum of characters  ${}^i\chi_\nu$ ,  $i \in \mathcal{J}$ . We can define functions  ${}^i\chi_{-\nu} : S^\nu \rightarrow \mathbb{G}_a$  so that

$$\chi_{-\nu}^\nu = \sum_{i \in \mathcal{J}} {}^i\chi_{-\nu}^\nu.$$

Let  $\chi_{-\nu, M_i}$  and  $\chi_{-\nu, M_i}^\nu$  be the corresponding objects for  $M_i$ .

**7.3.7. Lemma.**  ${}^i\chi_{-\nu}^\nu = \chi_{-\nu, M_i}^\nu \circ p_i^\nu$ .

Combining this lemma with Proposition 7.3.5 and the fact that Proposition 7.1.7 is true for  $M_i$  (proved in Sect. 7.3.1), we obtain:

**7.3.8. Corollary.** *If  $\nu \neq w_0(\lambda)$ , then for each component  $K$  of  $\text{Gr}^\lambda \cap S^\nu$ , there exists  $i \in \mathcal{J}$ , such that the restriction of  ${}^i\chi_{-\nu}^\nu$  to  $K(\overline{\mathbb{F}}_q)$  is non-constant.*

**7.3.9.** The subgroup  $T \subset T(\hat{\mathcal{O}}_x)$  acts on  $\text{Gr}$ . This action preserves both  $S^\nu$  and  $\text{Gr}^\lambda$ . Since  $T$  is connected,  $K$  is preserved as well.

The group  $T$  also acts on  $N(\hat{\mathcal{K}}_x)$  by conjugation, and therefore on the set of  $N(\hat{\mathcal{K}}_x)$ -characters: for  $\tau \in T$ ,  $\chi \rightarrow \chi^\tau$ , where  $\chi^\tau(n) = \chi(\tau \cdot n \cdot \tau^{-1})$ . We have:

$$({}^i\chi_{-\nu})^\tau = \check{\alpha}_i(\tau) \cdot {}^i\chi_{-\nu}.$$

Without loss of generality we can assume that for all  $j \in \mathcal{J}$  the function  ${}^j\chi_{-\nu}^\nu$  satisfies  ${}^j\chi_{-\nu}^\nu(\nu(t)) = 0$ . Then, since the point  $\nu(t) \in \text{Gr}$  is  $T$ -stable, we obtain:

$$(7.7) \quad ({}^i\chi_{-\nu}^\nu)^\tau = \check{\alpha}_i(\tau) \cdot {}^i\chi_{-\nu}^\nu.$$

Now we can prove Proposition 7.1.7. It suffices to show that on the level of  $\overline{\mathbb{F}}_q$ -points, the restriction of  $\chi_{-\nu}^\nu$  to a component  $K$  as above is non-constant. Suppose, to the contrary, that it is constant.

Letting  $i$  be as in Corollary 7.3.8, we obtain:

$$(7.8) \quad {}^i\chi_{-\nu}^\nu|_K = - \sum_{j \neq i} {}^j\chi_{-\nu}^\nu|_K + \text{const}$$

Let  $\tau$  be an element of  $T$  satisfying  $\check{\alpha}_i(\tau) \neq 1$ ,  $\check{\alpha}_j(\tau) = 1$  for  $j \neq i$ . Apply  $\tau$  to both sides of (7.8). According to (7.7) we obtain:

$$\check{\alpha}_i(\tau) \cdot {}^i\chi_{-\nu}^\nu|_K = \sum_{j \neq i} {}^j\chi_{-\nu}^\nu|_K + \text{const},$$

which, together with (7.8), contradicts the fact that  ${}^i\chi_{-\nu}^\nu|_K$  is not constant.

This completes the proof of Proposition 7.1.7.

## 8. PROOF OF THE CONJECTURE FROM [9]

### 8.1. The statement of the conjecture and its generalization.

**8.1.1.** As was explained in the Introduction, one of the main motivations for this paper was the following:

**8.1.2. Conjecture.** ([9]) *The cohomology  $H_c^k(\overline{\text{Gr}}^\lambda \cap S^\nu, \mathcal{A}_\lambda|_{\overline{\text{Gr}}^\lambda \cap S^\nu} \otimes \chi_0^\nu|_{\overline{\text{Gr}}^\lambda \cap S^\nu}^*(\mathcal{J}_\psi))$  vanishes unless  $k = \langle 2\nu, \check{\rho} \rangle$  and  $\nu = \lambda$ . In the latter case it is canonically isomorphic to  $\mathbb{Q}_\ell$ .*

In this section we will prove Theorem 1, which is a generalization of this conjecture. Recall that Theorem 1 states that for  $\lambda \in \Lambda^{++}$  and  $\mu, \nu \in \Lambda$  with  $\mu + \nu \in \Lambda^{++}$  the cohomology

$$(8.1) \quad H_c^k(\overline{\text{Gr}}^\lambda \cap S^\nu, \mathcal{A}_\lambda|_{\overline{\text{Gr}}^\lambda \cap S^\nu} \otimes \chi_\mu^\nu|_{\overline{\text{Gr}}^\lambda \cap S^\nu}^*(\mathcal{J}_\psi))$$

vanishes unless  $k = \langle 2\nu, \check{\rho} \rangle$  and  $\mu \in \Lambda^{++}$ . In the latter case, this cohomology identifies canonically with  $\text{Hom}_{L_G}(V^\lambda \otimes V^\mu, V^{\mu+\nu})$ .

**8.1.3.** As explained below, this theorem follows from Corollary 5.4.3 and Theorem 3(2) as a combination of two facts. First, we can readily recognize the above cohomology groups as the stalks of the LHS of (5.6) (up to replacing  $\lambda$  by  $-w_0(\lambda)$ ). Second, we can compute the stalks of the RHS of (5.6) using Theorem 3.

**8.1.4.** Note that Conjecture 8.1.2 is a special case of Theorem 1, when we set  $\mu = 0$ .

Now let  $\mu$  be very large compared to  $\lambda$  and  $\nu$ . In this case the function  $\chi_\mu^\nu : \overline{\text{Gr}}^\lambda \cap S^\nu \rightarrow \mathbb{G}_a$  is constant, i.e.,  $\chi_\mu^\nu|_{\overline{\text{Gr}}^\lambda \cap S^\nu}^*(\mathcal{J}_\psi)$  is isomorphic to  $\mathbb{Q}_\ell$ . Note also that in the case when  $\lambda, \nu \ll \mu$ , the vector space  $\text{Hom}_{L_G}(V^\mu \otimes V^\lambda, V^{\nu+\mu})$  can be naturally identified with the dual of the  $\nu$ -weight space  $V^\lambda(\nu)$ . Thus, Theorem 1 yields the following result which was previously proved by Mirković and one of the authors by different methods:

**Theorem.** *The cohomology  $H_c^k(\overline{\text{Gr}}^\lambda \cap S^\nu, \mathcal{A}_\lambda|_{\overline{\text{Gr}}^\lambda \cap S^\nu})$  vanishes unless  $k = \langle 2\nu, \check{\rho} \rangle$  and, in the latter case, it is isomorphic to  $V^\lambda(\nu)^*$ .*

Finally, note that Corollary 7.1.8 is also a special case of Theorem 1 with  $\mu = -\nu$  (see also Remark 2 below).

**8.1.5.** Let us make several remarks concerning the structure of Theorem 1.

*Remark 1.*

The vanishing part of Theorem 1 is obvious when  $\mu$  is non-dominant. Indeed, the group  $N(\hat{\mathcal{O}}_x)$  acts on  $\overline{\text{Gr}}^\lambda \cap S^\nu$  and the complex  $\mathcal{A}_\lambda|_{\overline{\text{Gr}}^\lambda \cap S^\nu} \otimes \chi_\mu^\nu|_{\overline{\text{Gr}}^\lambda \cap S^\nu}^*(\mathcal{J}_\psi)$  is  $(N(\hat{\mathcal{O}}_x), \chi_\mu)$ -equivariant, while  $\chi_\mu|_{N(\hat{\mathcal{O}}_x)}$  is non-trivial if  $\mu$  is non-dominant.

*Remark 2.*

It follows immediately from Proposition 7.1.3 that

$$H_c^k(\overline{\text{Gr}}^\lambda \cap S^\nu, \mathcal{A}_\lambda|_{\overline{\text{Gr}}^\lambda \cap S^\nu} \otimes \chi_\mu^\nu|_{\overline{\text{Gr}}^\lambda \cap S^\nu}^*(\mathcal{J}_\psi)) = 0$$

if  $k > \langle 2\nu, \check{\rho} \rangle$  and that for  $k = \langle 2\nu, \check{\rho} \rangle$  it is isomorphic to

$$H_c^{\langle \lambda+\nu, 2\check{\rho} \rangle}(\overline{\text{Gr}}^\lambda \cap S^\nu, \chi_\mu^\nu|_{\overline{\text{Gr}}^\lambda \cap S^\nu}^*(\mathcal{J}_\psi)).$$

*Remark 3.*

Assume now that  $G$  is defined and split over  $\mathbb{F}_q$ . Then the cohomology

$$H_c^{\langle \nu, 2\check{\rho} \rangle}(\overline{\text{Gr}}^\lambda \cap S^\nu, \mathcal{A}_\lambda|_{\overline{\text{Gr}}^\lambda \cap S^\nu} \otimes \chi_\mu^\nu|_{\overline{\text{Gr}}^\lambda \cap S^\nu}^*(\mathcal{J}_\psi))$$

is also defined over  $\mathbb{F}_q$ . The action of the Frobenius on the cohomology is easy to recover:

The previous remark implies that  $\text{Fr}$  acts on it by  $q^{\langle \nu, 2\check{\rho} \rangle}$ , i.e., as on  $\overline{\mathbb{Q}}_\ell(-\langle \nu, 2\check{\rho} \rangle)$ .

*Remark 4.*

Consider the vector space  $\text{Hom}_{L_G}(V^\lambda \otimes V^{-w_0(\mu+\nu)}, V^{-w_0(\mu)})$ . The previous discussion implies that it can be naturally identified with

$$\text{Hom}_{L_G}(V^\lambda \otimes V^\mu, V^{\nu+\mu}) \simeq H_c^{\langle \lambda+\nu, 2\check{\rho} \rangle}(\text{Gr}^\lambda \cap S^\nu, \chi_\mu^\nu|_{\text{Gr}^\lambda \cap S^\nu}^*(\mathcal{J}_\psi)).$$

Therefore, it acquires a basis labeled by the set  $\{K\}$  of those irreducible components  $K$  of  $\text{Gr}^\lambda \cap S^\nu$ , for which  $\chi_\mu^\nu|_K$  is non-constant (see the proof of Corollary 7.1.8).

It follows from Theorem 5.2.6 that  $\text{Hom}_{L_G}(V^\lambda \otimes V^{-w_0(\nu+\mu)}, V^{-w_0(\mu)})$  admits another basis of geometric origin as follows. Recall the scheme  $\text{Conv}$  (cf. Sect. 5.2.3). For  $\lambda_1$  and  $\lambda_2$  denote by  $\text{Conv}^{\lambda_1, \lambda_2}$  the corresponding locally closed subscheme in  $\text{Conv}$ . It is a fibration over  $\text{Gr}^{\lambda_1}$  with a typical fiber isomorphic to  $\text{Gr}^{\lambda_2}$ . It is known ([18]) that the projection  $p_2$  is a semi-small map  $\text{Conv}^{\lambda_1, \lambda_2} \rightarrow \text{Gr}^{\lambda_1+\lambda_2}$ . Thus, to a triple of coweights  $\lambda_1, \lambda_2, \lambda_3$  one can attach the set  $K'(\lambda_1, \lambda_2, \lambda_3)$  of irreducible components of the fiber of  $\text{Conv}^{\lambda_1, \lambda_2}$  over a point of  $\text{Gr}^{\lambda_3}$ . Hence the set  $K'(\lambda_1, \lambda_2, \lambda_3)$  parametrizes a basis for  $\text{Hom}_{L_G}(V^{\lambda_1} \otimes V^{\lambda_2}, V^{\lambda_3})$ .

Put  $\lambda_1 = \lambda$ ,  $\lambda_2 = -w_0(\mu + \nu)$ ,  $\lambda_3 = -w_0(\mu)$ . One can show that the sets  $\{K'(\lambda_1, \lambda_2, \lambda_3)\}$  and  $\{K\}$  are in a natural bijection and that the corresponding bases of the vector space  $\text{Hom}_{L_G}(V^\lambda \otimes V^{-w_0(\mu+\nu)}, V^{-w_0(\mu)})$  coincide.

## 8.2. Proof of Theorem 1.

8.2.1. First, let us show that it is enough to consider the case when  $[G, G]$  is simply-connected.

Let  $G_1 \twoheadrightarrow G$  be a surjection of reductive groups such that its kernel is a (connected) torus. Then we have a surjection of the corresponding lattices  $\Lambda_1 \twoheadrightarrow \Lambda$ . Fix  $\lambda_1, \mu_1, \nu_1 \in \Lambda_1$  as in Theorem 1 and let  $\lambda, \mu, \nu$  be their images in  $\Lambda$ .

Starting with an arbitrary  $G$ , we can always find a group  $G_1$  as above such that  $[G_1, G_1]$  is simply-connected. Theorem 1 for  $G$  follows from Theorem 1 for  $G_1$  because of the following obvious fact:

8.2.2. **Lemma.** *The cohomology appearing in Theorem 1 for  $G_1$  and  $\lambda_1, \mu_1, \nu_1$  is naturally isomorphic to that of  $G$  and  $\lambda, \mu, \nu$ .*

8.2.3. Now we assume that  $[G, G]$  is simply-connected. Fix a data of  $\varpi$  satisfying  $\text{cond}_x(\varpi) = 0$ . As was explained before, Theorem 1 will be obtained by computing explicitly the restriction of  $\overline{\Psi}_\varpi^{x, \mu+\nu} \star \mathcal{A}_{-w_0(\lambda)}$  to  ${}_{x, \mu} \text{Bun}_N^{\mathcal{F}_T}$ . According to Remark 1 above, we can assume that  $\mu$  is dominant.

On the one hand, we claim that we have a canonical isomorphism

$$(8.2) \quad j_\mu^*(\overline{\Psi}_\varpi^{x, \mu+\nu} \star \mathcal{A}_{-w_0(\lambda)}) \simeq \Psi_\varpi^{x, \mu} \otimes \text{Hom}_{L_G}(V^\lambda \otimes V^\mu, V^{\nu+\mu}).$$

Indeed, by Corollary 5.4.3,

$$\overline{\Psi}_\varpi^{x, \mu+\nu} \star \mathcal{A}_{-w_0(\lambda)} \simeq \bigoplus_{\mu'} \overline{\Psi}_\varpi^{x, \mu'} \otimes \text{Hom}_{L_G}(V^{\mu'}, V^{-w_0(\lambda)} \otimes V^{\nu+\mu}).$$

But by Theorem 3(1),  $j_\mu^*(\overline{\Psi}_\varpi^{x,\mu'}) = 0$  unless  $\mu' = \mu$ . Therefore,

$$j_\mu^*(\overline{\Psi}_\varpi^{x,\mu+\nu} \star \mathcal{A}_{-w_0(\lambda)}) \simeq \Psi_\varpi^{x,\mu} \otimes \mathrm{Hom}(V^\mu, V^{-w_0(\lambda)} \otimes V^{\nu+\mu}) \simeq \Psi_\varpi^{x,\mu} \otimes \mathrm{Hom}(V^\lambda \otimes V^\mu, V^{\nu+\mu}).$$

8.2.4. On the other hand, let us compute  $j_\mu^*(\overline{\Psi}_\varpi^{x,\mu+\nu} \star \mathcal{A}_{-w_0(\lambda)})$  using the definition of the Hecke functors.

Recall the notation of Sect. 7.2.2. As  $\overline{\Psi}_\varpi^{x,\mu+\nu} \simeq \Psi_\varpi^{x,\mu+\nu}!$  (here we use Theorem 3(1) once again), using base change,  $j_\mu^*(\overline{\Psi}_\varpi^{x,\mu+\nu} \star \mathcal{A}_{-w_0(\lambda)})$  can be computed as follows:

We  $*$ -restrict  $\overline{\Psi}_\varpi^{x,\mu+\nu} \boxtimes \mathcal{A}_{-w_0(\lambda)}$  to  $Z^{\mu,\mu+\nu}$  and then apply the  $!$ -push-forward with respect to the map  $'h^- : Z^{\mu,\mu+\nu} \rightarrow {}_{x,\mu}\mathrm{Bun}_N^{\mathcal{F}_T}$ . Using Lemma 7.2.4(1), we identify  $Z^{\mu,\mu+\nu}$  with the fibration

$${}_\mu \widetilde{\mathcal{N}}_x^{\epsilon_\mu} \times_{N(\check{\mathcal{O}}_x)} S^\nu \xrightarrow{'h^-} {}_{x,\mu} \widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T}$$

and the support of  $(\overline{\Psi}_\varpi^{x,\mu+\nu} \boxtimes \mathcal{A}_{-w_0(\lambda)})|_{Z^{\mu,\mu+\nu}}$  is the substack

$${}_\mu \widetilde{\mathcal{N}}_x^{\epsilon_\mu} \times_{N(\check{\mathcal{O}}_x)} (\overline{\mathrm{Gr}}^\lambda \cap S^\nu)$$

of  $Z^{\mu,\mu+\nu}$ .

Moreover, according to formula (5.2) and Lemma 7.2.7(2), the restriction of

$$\overline{\Psi}_\varpi^{x,\mu+\nu} \boxtimes \mathcal{A}_{-w_0(\lambda)}$$

to this substack can be identified with the sheaf

$$(\Psi_\varpi^{x,\mu}) \boxtimes (\mathcal{A}_\lambda|_{\overline{\mathrm{Gr}}^\lambda \cap S^\nu} \otimes \chi_\mu^\nu|_{\overline{\mathrm{Gr}}^\lambda \cap S^\nu}^*(\mathcal{J}_\psi))[\langle \nu, 2\check{\rho} \rangle]$$

(note that  $\langle \nu, 2\check{\rho} \rangle = \dim({}_{x,\mu+\nu}\mathrm{Bun}_N^{\mathcal{F}_T}) - \dim({}_{x,\mu}\mathrm{Bun}_N^{\mathcal{F}_T})$  and that is why it enters into the above formula.)

Therefore, by the projection formula, we obtain that  $j_\mu^*(\overline{\Psi}_\varpi^{x,\mu+\nu} \star \mathcal{A}_{-w_0(\lambda)})$  is isomorphic to

$$\Psi_\varpi^{x,\mu} \otimes H_c^\bullet(\mathcal{A}_\lambda|_{\overline{\mathrm{Gr}}^\lambda \cap S^\nu} \otimes \chi_\mu^\nu|_{\overline{\mathrm{Gr}}^\lambda \cap S^\nu}^*(\mathcal{J}_\psi))[\langle \nu, 2\check{\rho} \rangle].$$

Comparing it with (8.2), we obtain the statement of Theorem 1.

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